First-Order Loss Functions vs. CVaR in Inventory

Clarifying the Relationship Between Expected Shortage and Conditional Value at Risk

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1. Definitions

Consider demand X as a continuous random variable with PDF $f(x)$, CDF $F(x)$, and inventory target level y. Let $\alpha = F(y)$ represent the non-stockout probability (service level), so $1-\alpha = \mathbb{P}(X > y)$ is the stockout probability.

- **First-Order Loss Function $L(y)$:** Measures the *unconditional expected shortage* (units lost or backordered):

$$L(y) = \mathbb{E}[(X-y)^+] = \int_y^\infty (x-y) f(x)\, dx \tag{1}$$

- **Conditional Value at Risk $\mathrm{CVaR}_\alpha(X)$:** Measures the *conditional expected demand* given that demand exceeds inventory level $y = \mathrm{VaR}_\alpha(X)$:

$$\mathrm{CVaR}_\alpha(X) = \mathbb{E}[X \mid X > y] = \frac{1}{1-\alpha}\int_y^\infty x f(x)\, dx \tag{2}$$

2. Mathematical Relationship

Expanding the integral definition of the first-order loss function $L(y)$:

$$L(y) = \int_y^\infty x f(x)\, dx - y\int_y^\infty f(x)\, dx \tag{3}$$

Recognizing that $\int_y^\infty f(x)\, dx = \mathbb{P}(X > y) = 1-\alpha$, we substitute the expression for $\mathrm{CVaR}_\alpha(X)$:

$$L(y) = (1-\alpha)\cdot \mathrm{CVaR}_\alpha(X) - y(1-\alpha) \tag{4}$$

Rearranging terms yields the precise analytical bridge:

$$\mathrm{CVaR}_\alpha(X) = y + \frac{L(y)}{1-\alpha} \quad \Longleftrightarrow \quad L(y) = (1-\alpha)\Big(\mathrm{CVaR}_\alpha(X) - y\Big)$$

Using the **Mean Excess Function** $e(y) = \mathbb{E}[X - y \mid X > y]$, we can express both measures succinctly:

$$L(y) = (1-\alpha)\cdot e(y), \qquad \mathrm{CVaR}_\alpha(X) = y + e(y) \tag{5}$$

3. Key Distinctions

Unconditional vs. Conditional Domain: $L(y)$ computes expected loss across the *entire* sample space (including $X \le y$ where shortage is zero). $\mathrm{CVaR}_\alpha(X)$ conditions strictly on the *tail event* $X > y$.

Physical Unit Interpretation: $L(y)$ measures expected volume of shorted demand per cycle (e.g., "we expect to short 5 units on average"). $\mathrm{CVaR}_\alpha(X)$ measures total expected demand during a stockout event (e.g., "when a stockout occurs, total demand averages 105 units").

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