The Newsvendor problem

analysis of the cost structure under normally distributed demand

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Abstract

Well-known derivation of a closed form solution for the expected total cost expression of the Newsvendor problem.

The Newsvendor problem

We consider the Newsvendor problem, this is a classical inventory control problem in which we consider a single item and a single stocking location. The aim is to control stock over a single period planning horizon under a single opportunity to replenish stocks at the beginning of the horizon. Without loss of generality, we assume the initial inventory to be zero. Consider an order quantity Q and a normally distributed demand d with mean μ and standard deviation σ. Let ϕ_d and Φ_d denote the probability density function and the cumulative distribution function of d, respectively; furthermore, we shall denote as ϕ and Φ the probability density function and the cumulative distribution function of a standard normally distributed random variable, respectively. The newsvendor faces "overage" costs of o dollars per unit left, if the end of period inventory is positive. Conversely, the newsvendor faces "underage" costs of u dollars per unit short, if the end of period inventory is negative. The aim is to find the optimal order quantity Q that minimizes expected total cost, $C(Q)$, composed by expected total overage and underage costs:

$$C(Q) = o \int_{-\infty}^{Q} (Q-t)\, \phi_d(t)\, dt + u \int_{Q}^{\infty} (t-Q)\, \phi_d(t)\, dt$$

- **Example**

```
μ = 5;
σ = 2;
o = 1;
u = 4;
NMinimize[o ∫_{-∞}^{q} (q - t) PDF[NormalDistribution[μ, σ], t] ⅆt +
  u ∫_{q}^{∞} (t - q) PDF[NormalDistribution[μ, σ], t] ⅆt, {q}]
{2.79962, {q → 6.68324}}
```

A well-known result in inventory theory is the following

$$C(Q) = (o+u)\,\sigma \left(\phi\!\left(\frac{Q-\mu}{\sigma}\right) - \left(1-\Phi\!\left(\frac{Q-\mu}{\sigma}\right)\right)\frac{Q-\mu}{\sigma}\right) + o(Q-\mu)$$

or alternatively

$$C(Q) = (o+u)\,\sigma\,\left(\Phi\left(\tfrac{Q-\mu}{\sigma}\right)\tfrac{Q-\mu}{\sigma} + \phi\left(\tfrac{Q-\mu}{\sigma}\right)\right) - u(Q-\mu)$$

■ Example

```
Q = 6.683242467145828;
N[(o + u) σ (PDF[NormalDistribution[0, 1], (Q - μ)/σ] -
    (1 - CDF[NormalDistribution[0, 1], (Q - μ)/σ]) (Q - μ)/σ) + o (Q - μ)]
N[(o + u) σ ( CDF[NormalDistribution[0, 1], (Q - μ)/σ] (Q - μ)/σ + PDF[NormalDistribution[0, 1], (Q - μ)/σ]) -
   u (Q - μ)]
2.79962
2.79962
```

In the rest of this note we shall analytically derive the second of these expressions, the first can be derived in a similar fashion.

Consider the expected total cost

$$C(Q) = o\int_{-\infty}^{Q}(Q-t)\,\phi_d(t)\,dt + u\int_{Q}^{\infty}(t-Q)\,\phi_d(t)\,dt$$

and separate the overage component and the underage component. The overage is

$$o\int_{-\infty}^{Q}(Q-t)\,\phi_d\,(t)\,dt$$

the underage is

$$u\int_{Q}^{\infty}(t-Q)\,\phi_d(t)\,dt$$

the underage can be rewritten as

$$u\int_{-\infty}^{Q}(t-Q)\,\phi_d(t)\,dt - u(Q-\mu)$$

note that to derive the other expression discussed above, we must rewrite the overage cost in a similar fashion, rather than the underage cost, as here discussed.

Let us now adopt this latter expression in C(Q)

$$C(Q) = o\int_{-\infty}^{Q}(Q-t)\,\phi_d(t)\,dt + u\int_{-\infty}^{Q}(t-Q)\,\phi_d(t)\,dt - u(Q-\mu)$$

rewrite

$$C(Q) = (o+u)\int_{-\infty}^{Q}(Q-t)\,\phi_d(t)\,dt - u(Q-\mu)$$

By noting that

$$\int_{-\infty}^{Q}(t-Q)\,\phi_d(t)\,dt = \int_{-\infty}^{Q}\Phi_d(t)\,dt$$

we rewrite

$$C(Q) = (o+u) \int_{-\infty}^{Q} \Phi_d(t)\, dt - u(Q-\mu)$$

We now standardize the above expression by using the standard normal distribution

$$C(Q) = (o+u)\ \sigma \int_{-\infty}^{\frac{Q-\mu}{\sigma}} \Phi(t)\, dt - u(Q-\mu)$$

integrate by parts the expression

$$C(Q) = (o+u)\,\sigma\left(\Phi\left(\frac{Q-\mu}{\sigma}\right)\frac{Q-\mu}{\sigma} - \int_{-\infty}^{\frac{Q-\mu}{\sigma}} t\,\phi(t)\, dt\right) - u(Q-\mu)$$

rewrite

$$C(Q) = (o+u)\,\sigma\left(\Phi\left(\frac{Q-\mu}{\sigma}\right)\frac{Q-\mu}{\sigma} - \int_{-\infty}^{\frac{Q-\mu}{\sigma}} t\,\phi(t)\, dt\right) - u(Q-\mu)$$

$$C(Q) = (o+u)\,\sigma\left(\Phi\left(\frac{Q-\mu}{\sigma}\right)\frac{Q-\mu}{\sigma} + \phi\left(\frac{Q-\mu}{\sigma}\right)\right) - u(Q-\mu)$$

qed.

Example

```
Q = 6.683242467145828;
μ = 5;
σ = 2;
o = 1;
u = 4;
Print["C(Q)="];
```

$$N\left[o\int_{-\infty}^{Q}(Q-t)\,\text{PDF}[\text{NormalDistribution}[\mu,\sigma],t]\,dt+u\int_{Q}^{\infty}(t-Q)\,\text{PDF}[\text{NormalDistribution}[\mu,\sigma],t]\,dt\right]$$

```
Print["Overage="];
```

$$N\left[o\int_{-\infty}^{Q}(Q-t)\,\text{PDF}[\text{NormalDistribution}[\mu,\sigma],t]\,dt\right]$$

```
Print["Underage="];
```

$$N\left[u\int_{Q}^{\infty}(t-Q)\,\text{PDF}[\text{NormalDistribution}[\mu,\sigma],t]\,dt\right]$$

```
Print["Underage="];
```

$$N\left[u\int_{-\infty}^{Q}(Q-t)\,\text{PDF}[\text{NormalDistribution}[\mu,\sigma],t]\,dt-u\,(Q-\mu)\right]$$

```
Print["C(Q)="];
```

$$N\left[o\int_{-\infty}^{Q}(Q-t)\,\text{PDF}[\text{NormalDistribution}[\mu,\sigma],t]\,dt+u\int_{-\infty}^{Q}(Q-t)\,\text{PDF}[\text{NormalDistribution}[\mu,\sigma],t]\,dt-u\,(Q-\mu)\right]$$

$$N\left[o\int_{-\infty}^{Q}(Q-t)\,\text{PDF}[\text{NormalDistribution}[\mu,\sigma],t]\,dt+u\int_{-\infty}^{Q}(Q-t)\,\text{PDF}[\text{NormalDistribution}[\mu,\sigma],t]\,dt-u\,(Q-\mu)\right]$$

$$N\left[(o+u)\int_{-\infty}^{Q}(Q-t)\,\text{PDF}[\text{NormalDistribution}[\mu,\sigma],t]\,dt-u\,(Q-\mu)\right]$$

$$N\left[(o+u)\int_{-\infty}^{Q}\text{CDF}[\text{NormalDistribution}[\mu,\sigma],t]\,dt-u\,(Q-\mu)\right]$$

$$N\left[(o+u)\,\sigma\int_{-\infty}^{\frac{Q-\mu}{\sigma}}\text{CDF}[\text{NormalDistribution}[0,1],t]\,1\,dt-u\,(Q-\mu)\right]$$

$$N\left[(o+u)\,\sigma\left(\text{CDF}\left[\text{NormalDistribution}[0,1],\frac{Q-\mu}{\sigma}\right]\frac{Q-\mu}{\sigma}-\right.\right.$$

$$\int_{-\infty}^{\frac{Q-\mu}{\sigma}} t\, \text{PDF}[\text{NormalDistribution}[0,1], t]\, dt\Big) - u\,(Q-\mu)\Big]$$

$$N\left[(o+u)\,\sigma\left(\text{CDF}\left[\text{NormalDistribution}[0,1], \frac{Q-\mu}{\sigma}\right]\frac{Q-\mu}{\sigma} - \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\frac{Q-\mu}{\sigma}} t\, e^{-\frac{t^2}{2}}\, dt\right) - u\,(Q-\mu)\right]$$

$$N\left[(o+u)\,\sigma\left(\text{CDF}\left[\text{NormalDistribution}[0,1], \frac{Q-\mu}{\sigma}\right]\frac{Q-\mu}{\sigma} + \frac{1}{\sqrt{2\pi}}\, e^{-\frac{\left(\frac{Q-\mu}{\sigma}\right)^2}{2}}\right) - u\,(Q-\mu)\right]$$

$$N\left[(o+u)\,\sigma\left(\text{CDF}\left[\text{NormalDistribution}[0,1], \frac{Q-\mu}{\sigma}\right]\frac{Q-\mu}{\sigma} + \text{PDF}\left[\text{NormalDistribution}[0,1], \frac{Q-\mu}{\sigma}\right]\right) - u\,(Q-\mu)\right]$$

$$N\left[(o+u)\,\sigma\left(\text{PDF}\left[\text{NormalDistribution}[0,1], \frac{Q-\mu}{\sigma}\right] - \left(1-\text{CDF}\left[\text{NormalDistribution}[0,1], \frac{Q-\mu}{\sigma}\right]\right)\frac{Q-\mu}{\sigma}\right) + o\,(Q-\mu)\right]$$

C(Q) =

2.79962

Overage =

1.90652

Underage =

0.893101

Underage =

0.893101

C(Q) =

2.79962

2.79962

2.79962

2.79962

2.79962

2.79962

2.79962

2.79962

2.79962

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