On a CP approach to solve a MINLP inventory model

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Inventory Control

- Computation of optimal replenishment policies under demand uncertainty.

When to order?
How much to order?

Demand Uncertainty

Production

Inventory

Customers
Newsvendor problem

- We want to determine the **optimal quantity** of newspaper we should buy in the morning to meet a **daily uncertain demand** that follows a known distribution.

- Two well known approaches: minimize the expected total cost under
  - Service level constraint
  - Shortage cost
Newsvendor problem

- Problem parameters
  - Holding cost $h$
  - Demand distribution $g(d)$

Service level

- *Service level* $\alpha$

Pr\{\text{\(S \geq d\)}\} $\geq \alpha$

$G(S) \geq \alpha$

$S^* = G^{-1}(\alpha)$

$E[TC] = \int_0^S (S-t)g(t)dt + \int_{\epsilon}^{\infty} (t-S)g(t)dt$

Shortage cost

- *Shortage cost* $\mathcal{S}$

$z = F_{N(0,1)}^{-1}\left(\frac{s}{s+h}\right)$

$E[TC] = h \cdot G(S) - s \cdot (1 - G(S))$

$\frac{\partial}{\partial S} E[TC] = h \cdot g(S) + s \cdot g(S) \geq 0$

$G(S^*) = \frac{s}{s+h} (= \alpha)$

$E[TC](S^*) = (h+s)g_{N(0,1)}(z) \cdot \sigma$
Newsvendor problem under shortage cost scheme

Cost analysis

Let $\frac{S - \mu}{\sigma} = z_\beta$, then for any given $S$ such that $G_{N(0,1)}\left(\frac{S - \mu}{\sigma}\right) = \beta$

we proved that the expected total cost for the single period newsvendor problem can be computed as

$$E[TC](S) = h z_\beta \sigma + (h + s)\sigma [ g_{N(0,1)}(z_\beta) - (1 - \beta) z_\beta ]$$

In the particular case where $\beta = \left(\frac{s}{s + h}\right)$, the $E[TC]$ becomes

$$E[TC](S^*) = (h + s) g_{N(0,1)}(z) \cdot \sigma$$

and $z, S^*$ are computed as shown before:

$$z = G_{N(0,1)}^{-1} \left(\frac{s}{s + h}\right) \quad S^* = \mu + z \sigma = G_{N(0,1)}^{-1} \left(\frac{s}{s + h}\right)$$
Newsvendor problem under shortage cost scheme

- Cost analysis:

\[ E[TC](S) = h z_\beta \sigma + (h + s)\sigma \left[ g_{N(0,1)}(z_\beta) - (1 - \beta)z_\beta \right] \]
(R^n, S^n) policy

- Replenishment cycle policy (R,S)
  - effective in **reducing planning instability**.
  - Silver [Sil – 98] points out that this policy is appealing in several cases:
    - Items ordered from the same supplier (joint replenishments)
    - Items with resource sharing
    - Workload prediction
    - ...

- Dynamic (R,S) [Boo – 88]
  - Considers a non-stationary demand over an N-period planning horizon
(R^n, S^n) policy assumptions [Tar – 06]

- **Dynamic (R,S) [Boo – 88]**
  - Considers a non-stationary demand over an N-period planning horizon

- **Negative orders** are not allowed, if the actual stocks exceed the order-up-to-level for a review, this excess stock is carried forward and not returned to the supplier
\((R^n, S^n)\) policy under shortage cost scheme: stochastic programming model [Tar – 06]

\[
\min E(TC) = \int_{d_1} \int_{d_2} \cdots \int_{d_N} \sum_{i=1}^{N} (a\delta_t + vX_t + hI^+_t + sI^-_t) g_1(d_1) \cdots g_N(d_N) d(d_1) \cdots d(d_N)
\]  

subject to

\[X_t > 0 \Rightarrow \delta_t = 1\]  
\[I_t = \sum_{i=1}^{t} (X_i - d_i)\]  
\[I^+_t = \max(0, I_t)\]  
\[I^-_t = \min(0, I_t)\]  
\[X_t, I^+_t, I^-_t \in \mathbb{Z}^+ \cup \{0\}, \quad I_t \in \mathbb{Z}, \quad \delta_t \in \{0, 1\}\]

for \(t = 1 \ldots N\), where

- \(d_t\) : the demand in period \(t\), a normal random variable with PDF \(g_t(d_t)\),
- \(a\) : the fixed ordering cost,
- \(v\) : the proportional direct item cost,
- \(h\) : the proportional stock holding cost,
- \(s\) : the proportional shortage cost,
- \(\delta_t\) : a \(\{0, 1\}\) variable that takes the value of 1 if a replenishment occurs in period \(t\) and 0 otherwise,
- \(I_t\) : the inventory level at the end of period \(t\), \(-\infty < I_t < +\infty\), \(I_0 = 0\)
- \(I^+_t\) : the excess inventory at the end of period \(t\) carried over to the next period,
- \(0 \leq I^+_t\),
- \(I^-_t\) : the shortages at the end of period \(t\), or magnitude of negative inventory
- \(0 \leq I^-_t\),
- \(X_t\) : the replenishment order placed and received in period \(t\), \(X_t \geq 0\).
The proposed non-stationary \((R^n, S^n)\) policy consists of a series of review times \(R^n\) and order-up-to-levels \(S^n\).

We now consider a review schedule which has \(m\) reviews over an \(N\)-period planning horizon with orders arriving at \(\{T_1, T_2, \ldots, T_m\}\), where \(T_j > T_{j-1}\). For convenience we always fix an order in period 1: \(T_1 = 1\).

In [Tar – 06] the order quantity \(X_{T_i}\) is expressed in term of a new variable \(S_t\) that may be interpreted as:

- The opening stock level for period \(t\), if there is no replenishment in this period \((t \neq T_i)\)
- The order-up-to-level for period \(t\) if a replenishment is scheduled in such a period \((t = T_i)\)
(R^n, S^n) policy under shortage cost scheme:

According to this transformation, by defining $D_{t_1,t_2} = \sum_{j=t_1}^{t_2} d_j$, the **expected total cost** in the former model is expressed as

$$\min \ E\{TC\} = \sum_{i=1}^{m} \left( a \delta_{T_i} + \sum_{i=T_{i}}^{T_{i+1}-1} E\{C_{T_i,t}\} \right) +$$

$$v I_N + v \int_{D_{1,N}} D_{1,N} \times g(D_{1,N})d(D_{1,N}), \tag{7}$$

The term $v \int_{D_{1,N}} D_{1,N} \times g(D_{1,N})d(D_{1,N})$ is constant and can therefore be ignored in the optimization model. $E\{C_{T_i,t}\}$ of Eq. (7) is defined as:

$$\int_{-\infty}^{S_{T_i}} h(S_{T_i} - D_{T_i,t}) g(D_{T_i,t})d(D_{T_i,t}) - \int_{S_{T_i}}^\infty s(S_{T_i} - D_{T_i,t}) g(D_{T_i,t})d(D_{T_i,t}). \tag{8}$$

that is the expected total cost of a **single-period newsvendor** problem:

$$E\{TC\} = h \int_{-\infty}^{S} (S - D) g(D)d(D) - s \int_{S}^{\infty} (S - D) g(D)d(D) \tag{9}$$
Multi-period newsvendor problem under shortage cost scheme

- Expected total cost of a multi-period newsvendor problem

\[
E\{TC\} = \sum_{k=i}^{j} \left( h \int_{-\infty}^{S} (S - d_{i,k}) g_{i,k}(d_{i,k}) \, d(d_{i,k}) - s \int_{S}^{\infty} (S - d_{i,k}) g_{i,k}(d_{i,k}) \, d(d_{i,k}) \right) \quad (13)
\]

The considerations in the former sections refer to a single-period problem, but they can be easily extended to a replenishment cycle \( R(i, j) \) that covers the period span \( i, \ldots, j \). The demand in each period is normally distributed with PDF \( g_i(d_j), \ldots, g_j(d_j) \). The cost for the multiple periods’ replenishment cycle, when ordering costs are neglected, can be expressed as
Multi-period newsvendor problem under shortage cost scheme

- By using the closed form expression already presented, the summation becomes:

$$\sum_{k=i}^{j} (h z_{\alpha(i,k)} \sigma_{i,k} + (h + s) \sigma_{i,k} [\phi(z_{\alpha(i,k)}) - (1 - \alpha(i,k)) z_{\alpha(i,k)}])$$

(15)

since the sum of convex functions is a convex function, this expression is convex.

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Three periods holding and shortage cost as a function of the opening inventory level $S$. The demand is normally distributed in each period with mean respectively 150, 100, 200, the coefficient of variation is 0.1. Holding cost is 1, shortage cost is 10.
Multi-period newsvendor problem under shortage cost scheme

The cost for a replenishment cycle can be expressed as:

\[ C(S_i, i, j) = a + \sum_{k=i}^{j} (h \alpha(i,k) \sigma_i,k + (h + s) \sigma_i,k[\phi(\alpha(i,k)) - (1 - \alpha(i,k))\alpha(i,k)]) \]

(17)

- **Upper bound** for opening-inventory-levels:

  we optimize the convex cost of \( R(1, N) \), this will produce a buffer stock \( b(1, N) \). Then for each period \( t \in \{1, ..., N\} \),

  \[ \max(S_t) = \sum_{t}^{N} \hat{d}_t + b(1, N). \]

- **Lower bound** for closing-inventory-levels:

  we consider the buffer stock \( b(\hat{i}, j) \) required to optimize the convex cost of each replenishment cycle \( R(i, j) \) considered independently on the others. The lower bound is the minimum of these values for \( j \in \{1, ..., N\} \) and \( i \in \{1, ..., j\} \).
(R^n, S^n) policy under shortage cost scheme: deterministic equivalent model

- A deterministic equivalent [Bir – 97] CP formulation is:

\[
\min \ E\{TC\} = C
\]

subject to

\[
\text{obj}\text{Constraint}\ (C, \bar{I}_1, \ldots, \bar{I}_N, \delta_1, \ldots, \delta_N, d_1, \ldots, d_N, a, h, s) \tag{19}
\]

and for \( t = 1 \ldots N \)

\[
\bar{I}_t + \bar{d}_t - \bar{I}_{t-1} \geq 0 \tag{20}
\]
\[
\bar{I}_t + \bar{d}_t - \bar{I}_{t-1} > 0 \Rightarrow \delta_t = 1 \tag{21}
\]
\[
\bar{I}_t \in \mathbb{Z}, \quad \delta_t \in \{0, 1\} \tag{22}
\]

The objective function (18) minimizes the expected total cost over the given planning horizon. \text{obj}\text{Constraint}(\cdot) dynamically computes buffer stocks and it assigns to \( C \) the expected total cost related to a given assignment for replenishment decisions, depending on the demand distribution in each period and on the given combination for problem parameters \( a, h, s \).
(\(R^n, S^n\)) policy under shortage cost scheme: \(\text{objConstraint}(\ldots)\)

- Propagation

\[
R(i, j) \quad \delta_i = 1 \quad \delta_{k \in \{i+1, \ldots, j\}} = 0 \quad \delta_{j+1} = 1
\]

\[
C(S_i, i, j) = a + \sum_{k=i}^{j} (h z_{\alpha(i, k)} \sigma_{k,k} + (h + s) \sigma_{i,k} [\phi(z_{\alpha(i, k)}) - (1 - \alpha(i, k)) z_{\alpha(i, k)}])
\]

(17)
(R^n, S^n) policy under shortage cost scheme: \textit{objConstraint}(\ldots)

- **Propagation**
  - Inventory conservation constraint \textbf{satisfied}:
    - \textit{Inventory conservation constraint} violated:
(\(R^n, S^n\)) policy under shortage cost scheme: 
\(\text{objConstraint}(\ldots)\)

- Propagation
  - Inventory conservation constraint \textbf{violated}:

\[\begin{align*}
\text{stocks} & \quad \text{period} \\
 i | k | j & \quad b(i,k) \quad b(k+1,j)
\end{align*}\]
(R^n, S^n) policy under shortage cost scheme: 

\textit{objConstraint(…)}

- Propagation
  - Inventory conservation constraint \textbf{violated}:

\[ \text{Propagation} \]

\[ \text{Inventory conservation constraint violated:} \]

\[ \text{ stocks } \]

\[ \text{period} \]

\[ \text{stocks} \]

\[ \text{period} \]

\[ \text{stocks} \]

\[ \text{period} \]

\[ \text{stocks} \]

\[ \text{period} \]
(R^n, S^n) policy under shortage cost scheme: Comparison: CP – MIP approach

- We now compare for a set of instances the solution obtained with our CP approach and the one provided by the MIP approach in [Tar – 06]

- We consider the following normally distributed demand over an 8-period planning horizon:

<table>
<thead>
<tr>
<th>Period</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>d_t</td>
<td>200</td>
<td>100</td>
<td>70</td>
<td>200</td>
<td>300</td>
<td>120</td>
<td>50</td>
<td>100</td>
</tr>
</tbody>
</table>

Expected demand values
(R^n, S^n) policy under shortage cost scheme:
Comparison: CP – MIP approach

- Deterministic problem [Wag – 58]:

\[ h = 1, \ a = 250, \ s = 10, \ v = 0, \ \tau = 0.0 \]
(R^n, S^n) policy under shortage cost scheme: Comparison: CP – MIP approach

- Stochastic problem. Instance 1 [Tar – 06]:

\[ h = 1, \ a = 250, \ s = 10, \ v = 0, \ \tau = 0.1 \]
(R^n, S^n) policy under shortage cost scheme: Comparison: CP – MIP approach

- Stochastic problem. Instance 2 [Tar – 06]:

\[ h = 1, a = 250, s = 10, v = 0, \tau = 0.2 \]
(R^n, S^n) policy under shortage cost scheme:
Comparison: CP – MIP approach

- Stochastic problem. Instance 3 [Tar – 06]:

![Graph showing inventory levels over periods]

\[ h = 1, \ a = 350, \ s = 50, \ v = 0, \ \tau = 0.3 \]
(R^n, S^n) policy under shortage cost scheme: Comparison: CP – MIP approach

- Stochastic problem. Instance 4 [Tar – 06]:

\[ h = 1, a = 350, s = 50, v = 15, \tau = 0.3 \]
(R^n, S^n) policy under shortage cost scheme: CP approach, extensions

- Dedicated **cost-based filtering** techniques (see [Foc – 99]) can be developed (work submitted to Annals of OR).

- In [Tar – 07] we already presented a similar filtering method under a service level constraint [Tar – 05, Tar – 04].
  - Dynamic programming relaxation [Tar – 96].

- Applying the same technique under a shortage cost scheme requires **additional insights**, similar to the ones presented in this work, about the convex cost structure of the problem.

- Similar techniques let us solve instances with **planning horizons up to 50 periods typically in less than a second** for the service level case [Tar – 09].
We presented a CP approach to compute \((R^n, S^n)\) policy parameters under nonstationary demand and a shortage cost scheme.

We compared our approach against a previously published MIP-based approximation method, which is typically faster than the pure CP approach.

Using a set of problem instances we showed that a piecewise approximation with seven segments usually provides good quality solutions, while using less segments can yield poor quality solutions.
(R^n, S^n) policy under shortage cost scheme:

References