Solving the Newsvendor Problem under Partial Demand Information

Roberto Rossi¹  Steven D Prestwich²  S Armagan Tarim³  Brahim Hnich⁴

¹Wageningen University, The Netherlands
²University College Cork, Ireland
³Hacettepe University, Turkey
⁴Izmir University of Economics, Turkey

ROADEF 2011, Saint-Étienne, France
The Newsboy problem

inventory

0

single period

time

Newsboy problem
The Newsboy problem

Order quantity

![Diagram showing order quantity (Q) on the y-axis and time on the x-axis for a single period.](Image)
The Newsboy problem

Demand structure

- inventory
- random demand: \( d \)

Order quantity: \( Q \)

Single period

Time
The Newsboy problem

Demand structure

inventory

pmf: $g(d)$ 0.8

order quantity: $Q$

random demand: $d$

0

single period

time


The Newsboy problem

Demand structure

\[ Q \]

inventory

pdf: \( g(d) \)

0.8

random demand: \( d \)

order quantity: \( Q \)

single period

time

0
The Newsboy problem

Cost structure

inventory

order quantity: \( Q \)

random demand: \( d \)

inventory holding costs

single period

time
The Newsboy problem

Cost structure

- inventory
- order quantity: $Q$
- random demand: $d$
- penalty costs
- single period
- time

The graph illustrates the cost structure over a single period with order quantity $Q$ on the y-axis and time on the x-axis. The random demand $d$ influences the inventory levels and associated costs.
Consider

- \( d \): a **one-period** random demand that follows a **probability distribution** \( f(d) \)
- \( h \): unit holding cost
- \( p \): unit penalty cost

Let

\[
g(x) = hx^+ + px^-,
\]

where \( x^+ = \max(x, 0) \) and \( x^- = -\min(x, 0) \).

The **expected total cost** is \( G(Q) = E[g(Q - d)] \),

where \( E[\cdot] \) denotes the expected value.
The Newsboy problem

Solution method

If $d$ is continuous, $G(Q)$ is **convex**.

The optimal order quantity is

$$Q^* = \inf\{ Q \geq 0 : \Pr\{d \leq Q\} = \frac{p}{p + h}\}.$$
If $d$ is discrete (e.g. Poisson),

$$\Delta G(Q) = h - (h + p) \Pr\{d > j\}$$

is non-decreasing in $Q$.

$$Q^* = \min\{Q \in \mathbb{N}_0 : \Delta G(Q) \geq 0\}.$$
Solution method: example

Demand follows a Poisson distribution $\text{Poisson}(\lambda)$, with demand rate $\lambda = 50$.

Holding cost $h = 1$, penalty cost $p = 3$.

The optimal order quantity $Q^*$ is equal to 55 and provides a cost equal to 9.1222.
Partial demand information

**Unknown distribution parameter(s)**

Assume now that the **demand distribution** is known, but one or more **distribution parameters** are unknown.

The decision maker has access to a set of $M$ **past realizations of the demand**.

From these she has to estimate the **optimal order quantity** (or quantities) and the **associated cost**.
**Unknown distribution parameter(s)**

Poisson demand, probability mass function:

\[ P(X=k) \]

\[ \lambda = \begin{cases} 1, \ 4, \ 10 \end{cases} \]

\( \lambda \) has to be **estimated** from past realizations.
Point estimates of the parameter(s)

Point estimates of the unknown parameters may be obtained from the available samples by using:

- maximum likelihood estimators, or
- the method of moments.

Point estimates for the parameters are then used in place of the unknown demand distribution parameters to compute:

- the estimated optimal order quantity $\hat{Q}^*$, and
- the associated estimated expected total cost $G(\hat{Q}^*)$. 
**Point estimates: example**

*M* observed **past demand data** $d_1, \ldots, d_M$.

Demand follows a **Poisson distribution** $\text{Poisson}(\lambda)$, with demand rate $\lambda$.

We estimate $\lambda$ using the **maximum likelihood estimator** (sample mean):

$$\hat{\lambda} = \frac{1}{M} \sum_{i=1}^{M} \lambda_i.$$ 

The decision maker employs the distribution $\text{Poisson}(\hat{\lambda})$ **in place** of the actual unknown demand distribution.
Point estimates of the parameter(s)

**Point estimates: example**

Holding cost: \( h = 1 \); penalty cost: \( p = 3 \); observed past demand data: \{51, 54, 50, 45, 52, 39, 52, 54, 50, 40\}.

\[ \hat{\lambda} = 48.7, \hat{Q}^* = 53 \text{ and } G(\hat{Q}^*) = 9.0035. \]
The Bayesian approach infers the distribution of parameter $\lambda$ given some past observations $d$ by applying Bayes’ theorem as follows:

$$p(\lambda|d) = \frac{p(d|\lambda)p(\lambda)}{\int p(d|\lambda)p(\lambda)d\lambda}$$

where

$p(\lambda)$ is the prior distribution of $\lambda$, and

$p(\lambda|d)$ is the posterior distribution of $\lambda$ given the observed data $d$. 
Bayesian approach

The **prior distribution** describes **an estimate** of the likely values that the parameter $\lambda$ might take, without taking the data into account. It is based on **subjective assessment** and/or **collateral data**.

A number of methods for constructing “**non-informative priors**” have been proposed (i.e. maximum entropy). These are meant to reflect the fact that the decision maker **ignores** of the prior distribution.

If prior and posterior distributions are in the same family, then they are called **conjugate distributions**.
Bayesian approach [Hill, 1997]


He considers a number of distributions (Binomial, Poisson and Exponential) and **derives posterior distributions for the demand** from a set of given data.

He adoptes **uninformative priors** to express an initial state of **complete ignorance** of the likely values that the parameter might take.

By using the posterior distribution he obtains an **estimated optimal order quantity** and the respective **estimated expected total cost**.
Bayesian approach

Bayesian approach: example

Holding cost: $h = 1$; penalty cost: $p = 3$; observed past demand data: 
\{51, 54, 50, 45, 52, 39, 52, 54, 50, 40\}.

\[ \hat{Q}^* = 54 \text{ and } G(\hat{Q}^*) = 9.4764. \]
Drawbacks of existing approaches

**Only provide point estimates** of the order quantity and of the expected total cost.

**Do not quantify the uncertainty** associated with this estimate.

- How do we distinguish a case in which we only have 10 past observations vs a case with 1000 past observations?

The bayesian approach produces results that are “biased” by the selection of the prior; the posterior distribution may not satisfy Kolmogorov axioms.

An alternative approach

We propose a solution method based on confidence interval analysis [Neyman, 1937].

**Observation**

Since we operate under partial information, it may not be possible to uniquely determine “the” optimal order quantity and the associated exact cost.

We argue that a possible approach consists in determining a range of “candidate” optimal order quantities and upper and lower bounds for the cost associated with these quantities.

This range will contain the real optimum according to a prescribed confidence probability $\alpha$. 
An alternative approach

\[ Q^* = 55 \]

inventory

candidate order quantities

\[ d = \text{Poisson}(\lambda) \]
\[ \lambda = 50 \]
\[ G(Q^*) = 9.1222 \]
\[ M \text{ observations} \]
\[ \text{confidence} = \alpha \]
\[ \text{cost} = (c_{lb}, c_{ub}) \]
Consider a set of $M$ samples $d_i$ drawn from a random demand $d$ that is distributed according to a Poisson law with unknown parameter $\lambda$.

We construct a **confidence interval** for the unknown demand rate $\lambda$ as follows

$$
\lambda_{lb} = \min \{ \lambda | \Pr \{ \text{Poisson}(M\lambda) \geq \bar{d} \} \geq (1 - \alpha)/2 \},
\lambda_{ub} = \max \{ \lambda | \Pr \{ \text{Poisson}(M\lambda) \leq \bar{d} \} \geq (1 - \alpha)/2 \},
$$

where $\bar{d} = \sum_{i=0}^{M} d_i$.

A **closed form expression** for this interval has been proposed by Garwood [1936] based on the chi-square distribution.
Consider the set of 10 samples

\[\{51, 54, 50, 45, 52, 39, 52, 54, 50, 40\},\]

and \(\alpha = 0.9\).

The confidence interval for the unknown demand rate \(\lambda\) is

\[ (\lambda_{lb}, \lambda_{ub}) = (45.1279, 52.4896), \]

Note that, by chance, this interval covers the actual demand rate \(\lambda = 50\) used to generate the samples.
Poisson demand

Candidate order quantities

Let $Q^*_{lb}$ be the **optimal order quantity** for the Newsvendor problem under a $\text{Poisson}(\lambda_{lb})$ demand.

Let $Q^*_{ub}$ be the **optimal order quantity** for the Newsvendor problem under a $\text{Poisson}(\lambda_{ub})$ demand.

Since $\Delta G(Q)$ is **non-decreasing** in $Q$, according to the available information, **with confidence probability** $\alpha$, the optimal order quantity $Q^*$ is a **member** of the set \{ $Q^*_{lb}$, $\ldots$, $Q^*_{ub}$ \}.
Consider the set of 10 samples

\{51, 54, 50, 45, 52, 39, 52, 54, 50, 40\},

and \(\alpha = 0.9\).

The candidate order quantities are
Confidence interval for the expected total cost

For a given order quantity $Q$ we can prove that

$$G_Q(\lambda) = h \sum_{i=0}^{Q} \Pr\{\text{Poisson}(\lambda) = i\}(Q - i) + p \sum_{i=Q}^{\infty} \Pr\{\text{Poisson}(\lambda) = i\}(i - Q),$$

is convex in $\lambda$.

Upper ($c_{ub}$) and lower ($c_{lb}$) bounds for the cost associated with a solution that sets the order quantity to a value in the set $\{Q_{lb}^*, \ldots, Q_{ub}^*\}$ can be easily obtained by using convex optimization approaches to find the $\lambda^*$ that maximizes or minimizes this function over $(\lambda_{lb}, \lambda_{ub})$. 
Confidence interval for the expected total cost

for $Q \in \{Q_{lb}^*, \ldots, Q_{ub}^*\}$. 
Expected total cost: example

Consider the set of 10 samples

\[\{51, 54, 50, 45, 52, 39, 52, 54, 50, 40\}\],

and \(\alpha = 0.9\).

The upper and lower bound for the expected total cost are

\[d = \text{Poisson}(\lambda)\]
\[\lambda = 50\]
\[G(Q^*) = 9.1222\]
\[M \text{ observations }\]
\[\text{confidence} = \alpha\]
\[\text{cost} = (8.6, 14.6)\]
**Expected total cost: example**

Assume we decide to order 53 items, according to what a **MLE approach** suggests.

As we have seen, **MLE estimates** an expected total cost of 9.0035 (note that the real cost we would face is 9.3693).

If we compute $c_{lb} = 8.9463$ and $c_{ub} = 11.0800$, then we know that with $\alpha = 0.9$ confidence this interval **covers the real cost** we are going to face by ordering 53 units.

Similarly, the Bayesian approach **only prescribes** $\hat{Q}^* = 54$ and estimates $G(\hat{Q}^*) = 9.4764$ (real cost is 9.1530), while we know that $c_{lb} = 9.0334$ and $c_{ub} = 10.3374$. 
Consider the case in which unobserved lost sales occurred and the $M$ observed past demand data, $d_1, \ldots, d_M$, only reflect the number of customers that purchased an item when the inventory was positive.

The analysis discussed above can still be applied provided that the confidence interval for the unknown parameter $\lambda$ of the Poisson($\lambda$) demand is computed as

\[
\begin{align*}
\lambda_{lb} &= \min\{\lambda | \Pr\{\text{Poisson}(\hat{M}\lambda) \geq \bar{d} \} \geq (1 - \alpha)/2\}, \\
\lambda_{ub} &= \max\{\lambda | \Pr\{\text{Poisson}(\hat{M}\lambda) \leq \bar{d} \} \geq (1 - \alpha)/2\}.
\end{align*}
\]

where $\hat{M} = \sum_{j=1}^{M} T_j$, and $T_j \in (0, 1)$ denotes the fraction of time in day $j$ — for which a demand sample $d_j$ is available — during which the inventory was positive.
Binomial demand

$N$ customers enter the shop on a given day, the unknown purchase probability of the Binomial demand is $q \in (0, 1)$.

The analysis is similar to that developed for a Poisson demand.

Also in this case we prove that $G_Q(q)$ is convex in $q$.

Lost sales can be easily incorporated in the analysis.
Other distributions

Exponential demand

The interval of candidate order quantities can be easily identified.

The analysis on the expected total cost is complicated by the fact that $G_Q(\lambda)$ is not convex.

Extension to include lost sales is difficult.
A number of properties of $G_Q(\lambda)$ can be exploited to find upper and lower bounds for the expected total cost.
A number of properties of $G_Q(\lambda)$ can be exploited to find upper and lower bounds for the expected total cost.
We presented a confidence-based optimization strategy to the Newsboy problem with unknown demand distribution parameter(s).

We applied our approach to three maximum entropy probability distributions of the exponential family.

We showed the advantages of our approach over two existing strategies in the literature.

For two demand distributions we extended the analysis to include lost sales.
Future works

Consider other probability distributions (e.g. Normal, LogNormal etc.).

Further develop the analysis on lost sales for the Exponential distribution.

Apply confidence-based optimization to other stochastic optimization problems.
Questions