A mixed integer linear programming heuristic for computing nonstationary (s,S) policy parameters

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UNIVERSITY OF EDINBURGH
Business School

Deterministic Lot sizing

Lot sizing under time-varying demand

- We consider a **planning horizon** comprising N **periods** (**periodic review**)
- **demand rate** is given in the form d_j for $j = 1, \ldots, N$
- no shortage allowed
- the **entire requirement** of each period must be **available at the beginning of the period**
- orders are placed and **immediately received** at the beginning of a period (zero lead time)
- **no capacity restrictions**
- **fixed production/setup** cost K
- inventory **holding cost** $h/(\text{unit} \cdot \text{period})$ **charged on inventory level at the end of a period**, after demand has occurred

Example: the forecasts for the demand of an item are given below for the upcoming 5 months

Month	1	2	3	4	5
Demand rate (units/month)	34	45	65	56	87

The **fixed cost** for a replenishment is $K = 100$ \$.

Cost of **carrying item in inventory** is $h = 1$ \$/(unit · month).

Note that the **average demand** per month is $d = 57.4$ units/month.

Deterministic Lot sizing

Lot sizing under time-varying demand

Possible solution strategies:

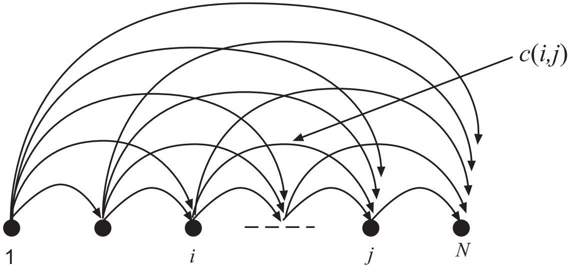
- solve the problem to optimality: **Wagner-Whitin algorithm**
- replace the time-varying demand with the average demand over the planning horizon and compute the associated **Economic Order Quantity** (EOQ)
- adopt a simple heuristic strategy: **Silver-Meal algorithm**

Deterministic Lot sizing

Lot sizing under time-varying demand: Wagner-Whitin algorithm

Solve the problem to optimality: **Wagner-Whitin algorithm**.[1]

Build a **directed acyclic graph** for all possible replenishment cycles (i, j), with associated cost $c(i, j)$.

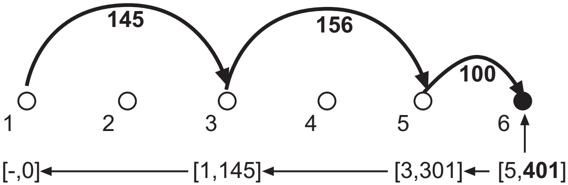


[1] H. M. Wagner and T. Whitin, "Dynamic version of the economic lot size model", Management Science, Vol. 5, pp. 89—96, 1958

Inventory Management

Lot sizing under time-varying demand: Wagner-Whitin algorithm

Obtain the optimal solution via dynamic programming (shortest path algorithm).



In this case **the optimal plan is**: to order in period 1 to cover demand in period 1 and 2, to order in period 3 to cover demand in period 3 and 4, and finally to order in period 5 to cover demand in period 5. The **cost of this plan** is 401 $.

Deterministic Lot sizing

Lot sizing under time-varying demand: EOQ strategy

Replace the **time-varying demand** with the **average demand** over the planning horizon and use the EOQ strategy.

Recall that the **EOQ** is

$$Q^* = \sqrt{\frac{2ad}{h}} = \sqrt{\frac{2 \cdot 100 \cdot 57.4}{1}} = 107.14 \text{ units}$$

and the **total relevant cost** per unit time is

$$c(Q^*) = \sqrt{2adh} = \sqrt{2 \cdot 100 \cdot 57.4 \cdot 1} = 107.14 \text{ \$/month}$$

since the planning horizon comprises $N = 5$ months, the **total cost over the horizon** is $107.14 \cdot 5 = 535.7$ Note that this is **not the actual cost** we will face, but only **a rough estimate**!

Deterministic Lot sizing

Lot sizing under time-varying demand: Silver-Meal algorithm I

Adopt a simple heuristic strategy: **Silver-Meal algorithm**.[2] This heuristics **selects the replenishment quantity** in such a way as **to myopically minimize the total relevant costs per unit time** for the duration of the replenishment cycle.

Month	1	2	3	4	5
Demand rate (units/month)	34	45	65	56	87

Cycle	$c(1,2)$	$c(1,3)$	$c(1,4)$
Cost per unit time	$100/1 = 100$	$145/2 = 72.5$	$275/3 = 91.66$

$c(1,4) > c(1,3)$, therefore we issue an order of 79 units in period 1, to cover demand in periods 1 and 2.

[2] E. A. Silver and H. C. Meal, "A Heuristic for Selecting Lot Size Quantities ...", Production and Inventory Management, 14(2):64—75, 1973

Deterministic Lot sizing

Lot sizing under time-varying demand: Silver-Meal algorithm II

Adopt a simple heuristic strategy: **Silver-Meal algorithm**.[3] This heuristics **selects the replenishment quantity** in such a way as **to myopically minimize the total relevant costs per unit time** for the duration of the replenishment cycle.

Month	1	2	3	4	5
Demand rate (units/month)	34	45	65	56	87

Cycle	$c(3,4)$	$c(3,5)$	$c(3,6)$
Cost per unit time	$100/1 = 100$	$156/2 = 78$	$330/3 = 110$

$c(3,6) > c(3,5)$, therefore we issue an order of 121 units in period 3, to cover demand in periods 3 and 4.

Finally, since period 5 is the last period in our horizon, we issue an order in period 5 to cover demand during this period.

Incidentally, this is the same plan obtained via the optimal approach.

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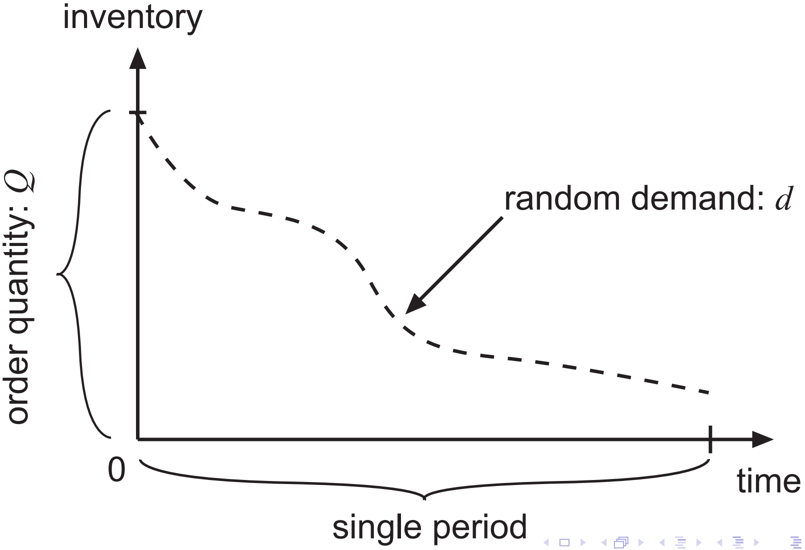
[3] E. A. Silver and H. C. Meal, "A Heuristic for Selecting Lot Size Quantities ...", Production and Inventory Management, 14(2):64—75, 1973

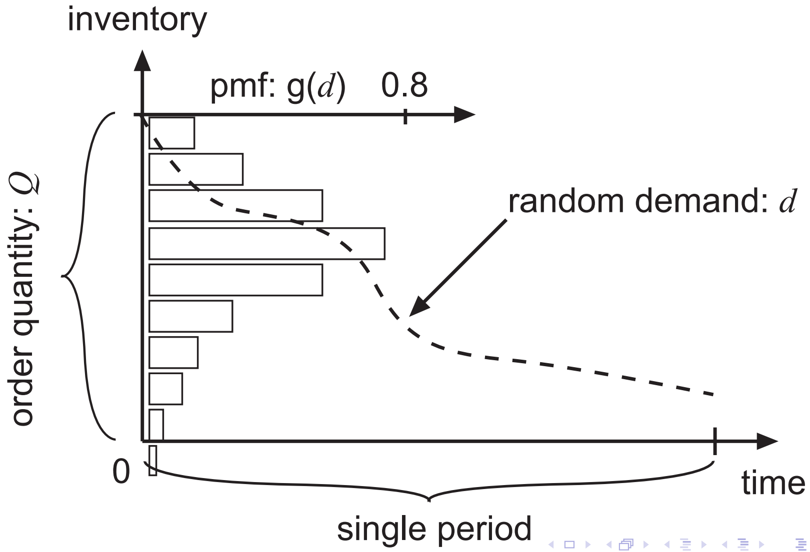
Stochastic lot-sizing

The newsboy problem



Stochastic lot-sizing

The newsboy problem



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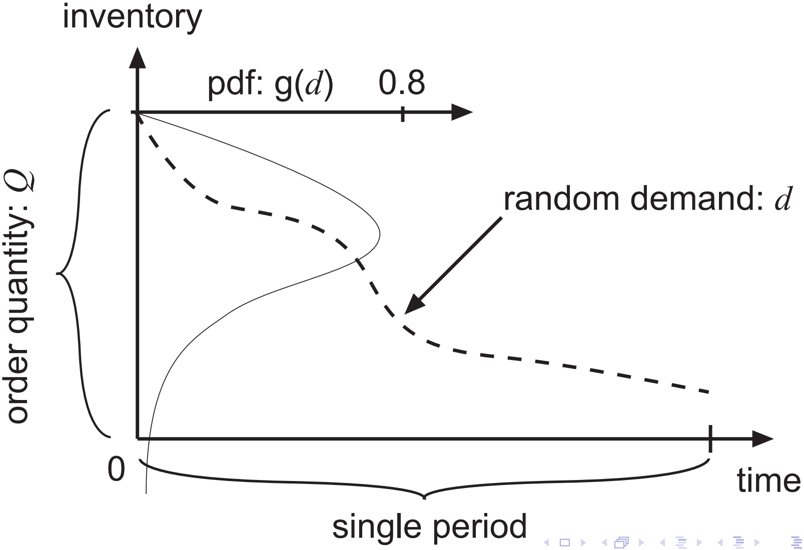

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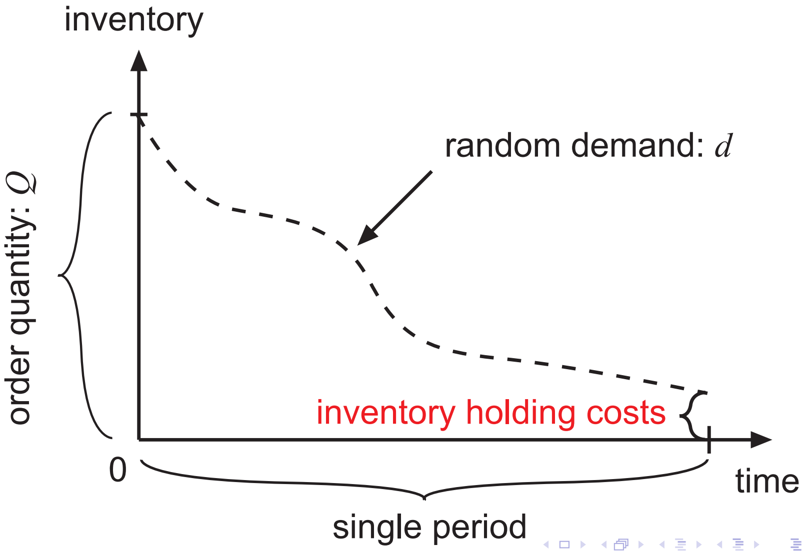

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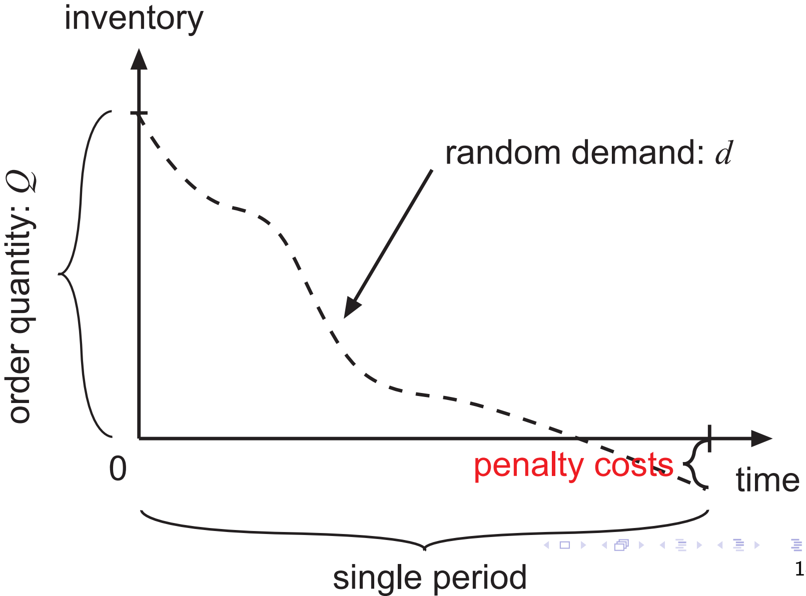


Stochastic lot-sizing

The newsboy problem

Consider

- d: a **one-period** random demand that follows a **probability distribution** $f(d)$
- h: unit **holding cost**
- p: unit **penalty cost**

Let I be the end of period inventory and

$$g(I) = hI^+ + pI^-,$$

where $I^+ = \max(I, 0)$ and $I^- = -\min(I, 0)$.

The **expected total cost** is $G(Q) = E[g(Q - d)]$, where $E[\cdot]$ denotes the expected value.

Stochastic lot-sizing

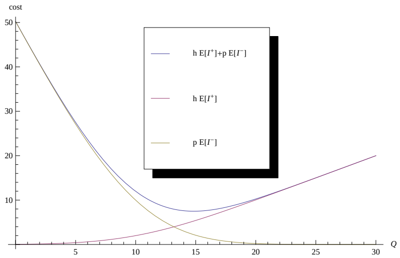
The newsboy problem

Define:

$$E[I^+] = E[\max(Q-d,0)]: \quad \text{complementary first order loss function}$$
$$E[I^-] = E[\max(d-Q,0)]: \quad \text{first order loss function}$$

The **expected total cost** comprises two separable components

$$G(Q) = E[g(Q-d)] = hE[I^+] + pE[I^-]$$



$d = \text{Normal}(10, 5)$
$h =$\$1
$p =$\$5

Stochastic lot-sizing

The first order loss function

Consider a continuous random variable ω with support over $\mathbb{R}$, probability density function $g_\omega(x) : \mathbb{R} \to (0,1)$ and cumulative distribution function $G_\omega(x) : \mathbb{R} \to (0,1)$.

The **first order loss function** can be rewritten as

$$\mathcal{L}(x,\omega) = \int_{-\infty}^{\infty} \max(t-x,0) g_\omega(t)\,dt = \int_{x}^{\infty} (t-x) g_\omega(t)\,dt. \quad (1)$$

The **complementary first order loss function** can be rewritten as

$$\widehat{\mathcal{L}}(x,\omega) = \int_{-\infty}^{\infty} \max(x-t,0) g_\omega(t)\,dt = \int_{-\infty}^{x} (x-t) g_\omega(t)\,dt. \quad (2)$$

Lemma

$\mathcal{L}(x,\omega)$ *and* $\widehat{\mathcal{L}}(x,\omega)$ *are convex in* x.

Stochastic lot-sizing

General framework

$$\min \mathsf{E}[\mathsf{TC}] = \int_{d_1}\int_{d_2}\dots\int_{d_N}\sum_{t=1}^{N}(K\delta_t + h\max(I_t,0) + vQ_t)\cdot$$
$$g_1(d_1)g_2(d_2)\dots g_N(d_N)\,d(d_1)d(d_2)\dots d(d_N)$$

subject to, for $t = 1,\dots N$

$$I_t = I_0 + \sum_{i=1}^{t}(Q_i - d_i)$$

$$\delta_t = \begin{cases} 1 & \text{if } Q_t > 0, \\ 0 & \text{otherwise} \end{cases}$$

$$Q_i \geq 0,\;\; \delta_t \in \{0,1\}$$

Stochastic lot-sizing

Penalty cost

$$\min \mathsf{E}[\mathsf{TC}] = \int_{d_1}\int_{d_2}\dots\int_{d_N}\sum_{t=1}^{N}$$
$$(K\delta_t + h\max(I_t,0) + p\max(-I_t,0) + vQ_t)\cdot$$
$$g_1(d_1)g_2(d_2)\dots g_N(d_N)\,d(d_1)d(d_2)\dots d(d_N)$$

subject to, for $t = 1,\dots N$

$$I_t = I_0 + \sum_{i=1}^{t}(Q_i - d_i)$$
$$\delta_t = \begin{cases} 1 & \text{if } Q_t > 0, \\ 0 & \text{otherwise} \end{cases}$$
$$Q_i \geq 0,\ \delta_t \in \{0,1\}$$

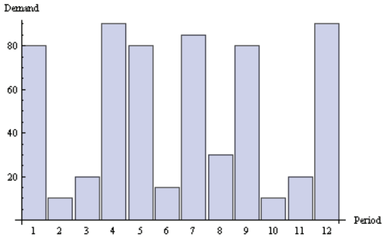
Problem parameters

We classify possible control strategies according to the **taxonomy** discussed in

J. H. Bookbinder and J. Y. Tan. Strategies for the probabilistic lot-sizing problem with service-level constraints. *Management Science*, 34:1096–1108, 1988

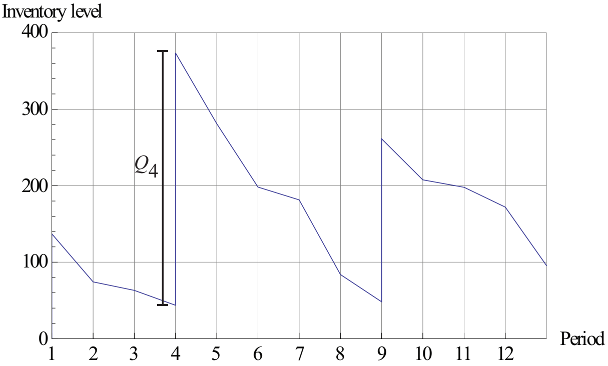
Normally distributed demand with coefficient of variation

$$c_v = \frac{\sigma_t}{\mu_t}$$



Static uncertainty

Charles R. Sox. Dynamic lot sizing with random demand and non-stationary costs.
Operations Research Letters, 20(4):155–164, May 1997



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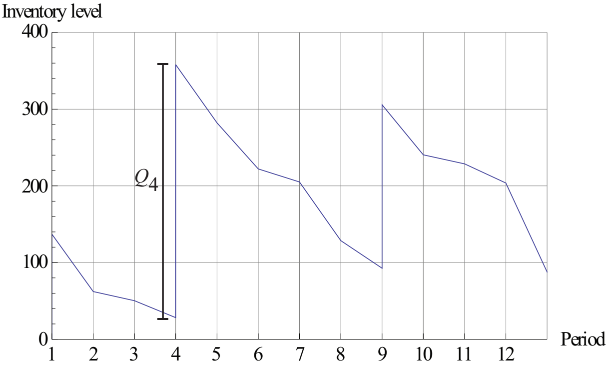

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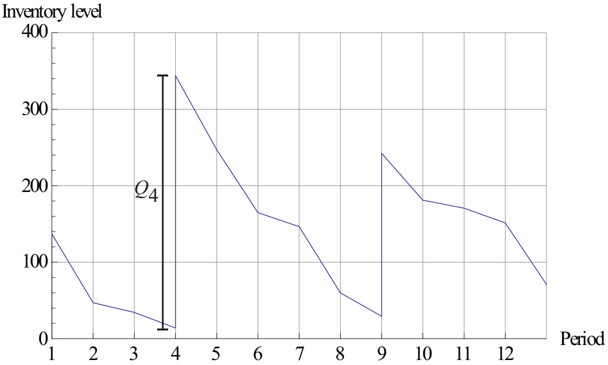

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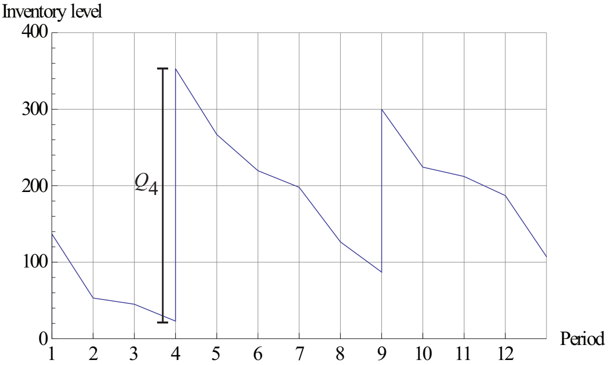

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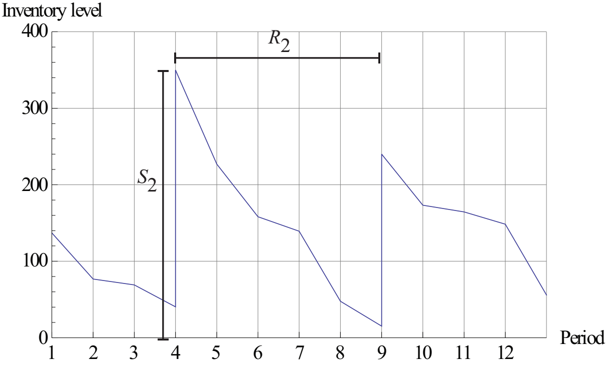

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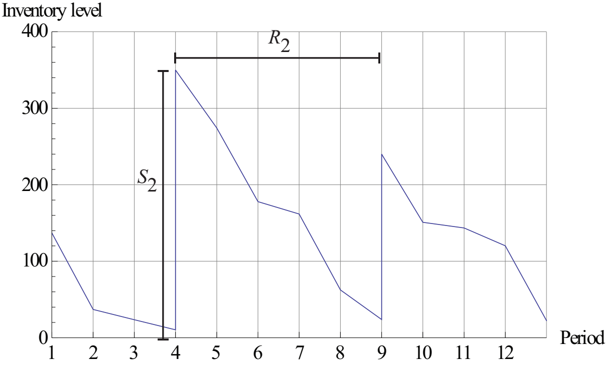

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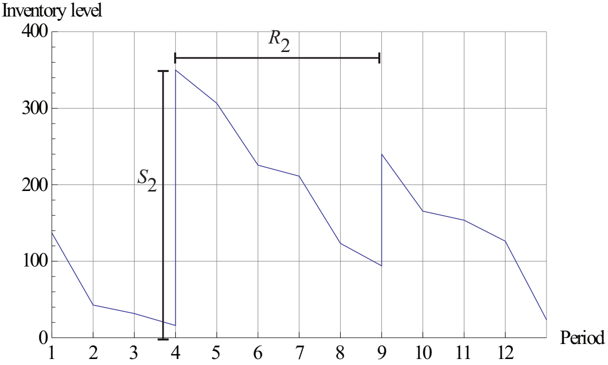

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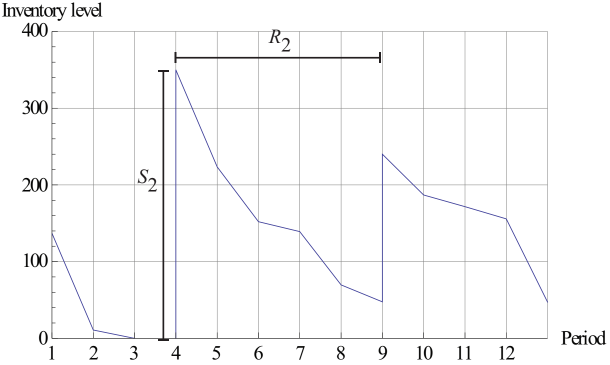
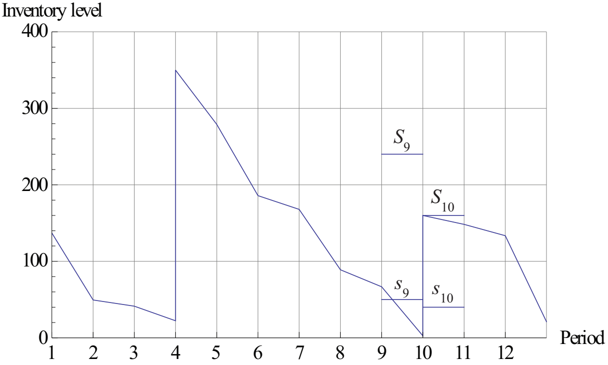


Dynamic uncertainty

H. Scarf. The optimality of (s, S) policies in the dynamic inventory problem.
In K. J. Arrow, S. Karlin, and P. Suppes, editors, *Mathematical Methods in the Social Sciences*, chapter 13, pages 196–202. Stanford University Press, Stanford, CA, 1960



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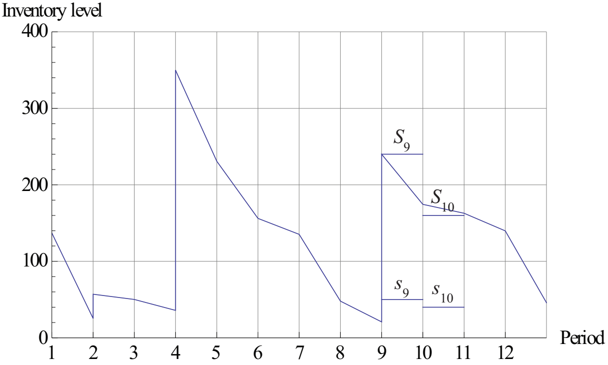

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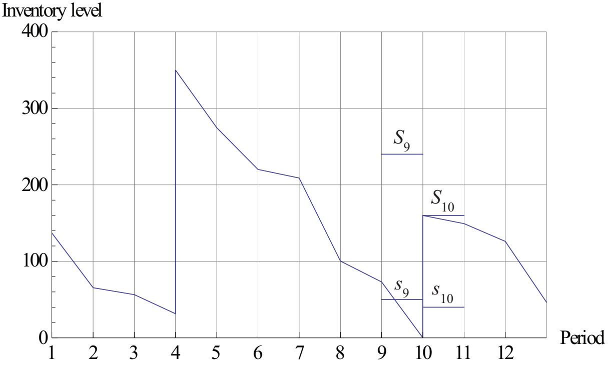

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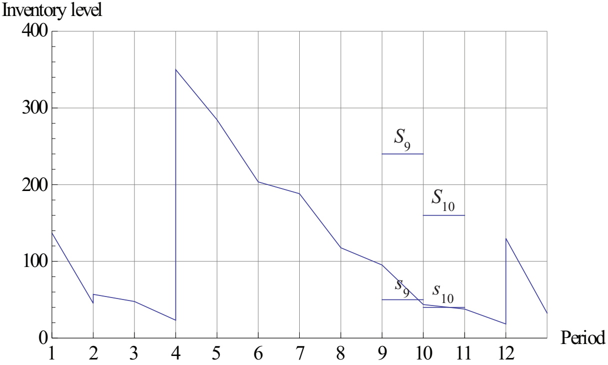


Dynamic uncertainty

Let $c(y)$ denote the **production/ordering cost**

$$c(y) = \begin{cases} K + vy & \text{if } \; y > 0 \\ 0 & \text{otherwise} \end{cases}$$

where y denote the stock level immediately after purchases are delivered.

The expected **holding and shortage cost** for a generic period are given by

$$L(y) = h \int_0^y (y - \omega) g(\omega) \mathrm{d}\omega + p \int_y^\infty (\omega - y) g(\omega) \mathrm{d}\omega$$

where $g_t(\cdot)$ denotes the probability density function of the demand in period t.

Let the initial inventory be x and $C_n(x)$ represent the **expected total cost** over the n-periods planning horizon **if provisioning is done optimally** then $C_n(x)$ satisfies

$$C_n(x) = \min_{y \geq x} \left\{ c(y - x) + L_n(y) + \int_0^\infty C_{n-1}(y - \omega) g_n(\omega) \mathrm{d}\omega \right\}$$

If $y_n(x)$ is the argument minimising the above functional equation, then $y_n(x) - x$ denotes the optimal initial purchase.

Dynamic uncertainty

The simple form of the optimal control policy stems from the study of the following function

$$G_n(y) = vy + L_n(y) + \int_0^\infty C_{n-1}(y-\omega)g_n(\omega)\mathrm{d}\omega$$

More specifically, Scarf proved that $G_n(y)$ is K-convex.

Definition

Let $K \geq 0$, and let $f(x)$ be a differentiable function, $f(x)$ is K-convex if

$$K + f(a+x) - f(x) - af'(x) \geq 0$$

for all positive a and all x.

This definition can be extended to a non differentiable function.

Dynamic uncertainty

We shall now illustrate graphically the notion of K-convexity on a simple numerical example.

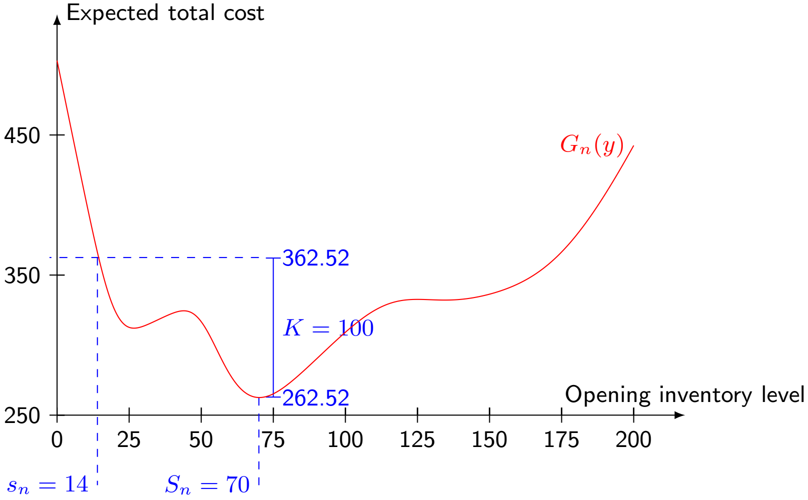
Consider a planning horizon of $n = 4$ period and a demand d_t normally distributed in each period t with mean $\mu_t \in \{20, 40, 60, 40\}$, for period $t = 1, \ldots, n$ respectively.

The standard deviation σ_t of the demand in period t is equal to $0.25\mu_t$.

Other problem parameters are $K = 100$, $h = 1$ and $p = 10$; to better conceptualise the example we let $v = 0$.

Dynamic uncertainty

We plot $G_n(y)$ for an initial inventory $y \in (0, 200)$.



Dynamic uncertainty

The fact that $G_n(y)$ is K-convex ensures that **ripples** in the nonlinear cost function **do not affect the existence of a unique reorder point** $s_n \leq S_n$, since their height will never exceed K.

It follows that, under general nonstationary settings, the **optimal policy** can be described via n pairs (s_i,S_i), where s_i denotes the reorder point and S_i the order-up-to-level for period i.

In practice, S_n denotes the **absolute minimum** of $G_n(y)$ and $s_n < S_n$ is the **unique value** such that $K + G_n(S_n) = G_n(s_n)$.

Unfortunately, computing optimal policy parameters (s_i,S_i) is **computationally expensive** (dynamic programming).

Dynamic uncertainty

Askin's heuristic for computing (s,S) policy parameters

Ronald G. Askin. A procedure for production lot sizing with probabilistic dynamic demand.
A I I E Transactions, 13(2):132–137, June 1981

Dynamic uncertainty

Bollapragada & Morton's heuristic for computing (s,S) policy parameters

Srinivas Bollapragada and Thomas E. Morton. A simple heuristic for computing nonstationary (s, s) policies.
Oper. Res., 47(4):576–584, April 1999

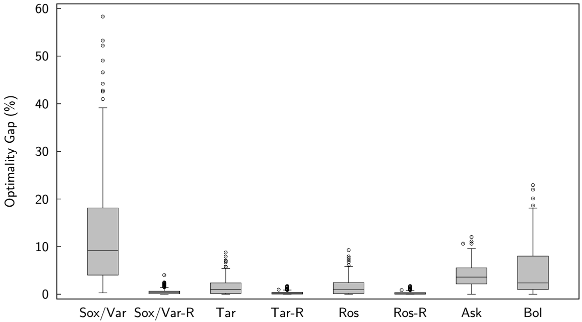
Yu-Sheng Zheng and A. Federgruen. Finding optimal (s, s) policies is about as simple as evaluating a single policy.
Operations Research, 39(4):654–665, 1991

Dynamic uncertainty

Rossi et al.'s heuristic for computing (s,S) policy parameters

Key insight: the static-dynamic uncertainty is a good proxy to the dynamic uncertainty policy from a cost perspective.

Gozdem Dural-Selcuk, Onur A. Kilic, S. Armagan Tarim, and Roberto Rossi. A comparison of methods for inventory problems with non-stationary stochastic demands.
Unpublished, 2014



Dynamic uncertainty

Rossi et al.'s heuristic for computing (s,S) policy parameters

Good news: we have just published a very effective method for computing static-dynamic uncertainty policies!

Roberto Rossi, Onur A. Kilic, and S. Armagan Tarim. Piecewise linear approximations for the static-dynamic uncertainty strategy in stochastic lot-sizing.
Omega, August 2014

Roberto Rossi, S. Armagan Tarim, Steven Prestwich, and Brahim Hnich. Piecewise linear lower and upper bounds for the standard normal first order loss function.
Applied Mathematics and Computation, 231:489–502, March 2014

The first order loss function

Piecewise linear approximations

We introduce a well-known inequality from stochastic programming

Peter Kall and Stein W. Wallace. *Stochastic Programming (Wiley Interscience Series in Systems and Optimization)*.
John Wiley & Sons, August 1994, p. 167.

Theorem (Jensen's inequality)

Consider a random variable ω with support Ω and a function $f(x,s)$, which for a fixed x is convex for all $s \in \Omega$, then

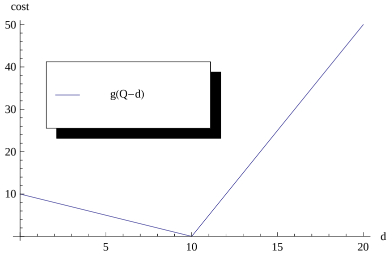
$$E[f(x,\omega)] \geq f(x, E[\omega]).$$

The first order loss function

The newsboy problem & Jensen's inequality

For a fixed Q, the **total cost** is convex for all values in the support of d.

$$g_Q(d) = g(Q-d) = h\max(Q-d,0) + p\max(d-Q,0))$$



$Q = 10$
$h =$ \$1
$p =$ \$5

The first order loss function

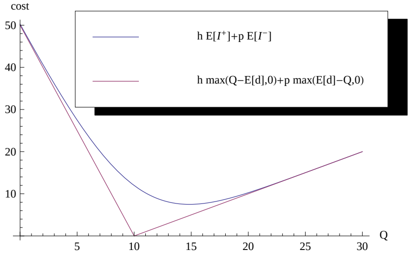
The newsboy problem & Jensen's inequality

Define:

$$E[I^+] = E[\max(Q-d,0)]: \quad \text{complementary first order loss function}$$
$$E[I^-] = E[\max(d-Q,0)]: \quad \text{first order loss function}$$

The **expected total cost** can be bounded from below as follows.

$$hE[I^+]+pE[I^-] \geq h\max(Q-E[d],0)+p\max(E[d]-Q,0) = g(Q-\mathsf{E}[d])$$



$d = \text{Normal}(10, 5)$
$\mathsf{E}[d] = 10$
$h =$ \$1
$p =$ \$5

The first order loss function

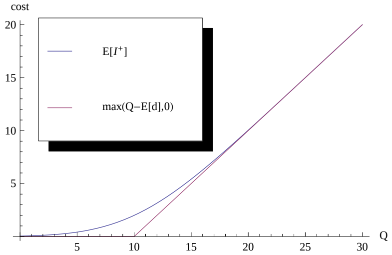
Bounding techniques

Define:

$$E[I^+] = E[\text{max}(Q-d,0)]: \quad \text{complementary first order loss function}$$

The **complementary first order loss function** can be bounded from below as follows.

$$E[I^+] \geq h \max(Q - E[d], 0)$$



$d = \text{Normal}(10, 5)$
$\text{E}[d] = 10$
$h =$\$1
$p =$\$5

The first order loss function

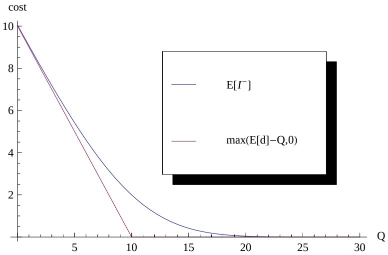
Bounding techniques

Define:

$$E[I^-] = E[\mathsf{max}(d-Q,0)]: \quad \text{first order loss function}$$

The **first order loss function** can be bounded from below as follows.

$$E[I^-] \geq \max(E[d]-Q,0)$$



$d =$ Normal$(10,5)$
E$[d] = 10$
$h =$\$1
$p =$\$5

The first order loss function

Bounding techniques

Let $g_\omega(\cdot)$ denote the probability density function of ω and consider a partition of the support Ω of ω into N disjoint compact subregions $\Omega_1, \ldots, \Omega_N$. We define, for all $i = 1, \ldots, N$

$$p_i = \Pr\{\omega \in \Omega_i\} = \int_{\Omega_i} g_\omega(t)\,dt$$

$$\mathsf{E}[\omega|\Omega_i] = \frac{1}{p_i}\int_{\Omega_i} t g_\omega(t)\,dt$$

Theorem

$$E[f(x,\omega)] \geq \sum_{i=1}^{N} p_i f(x, E[\omega|\Omega_i])$$

The first order loss function

Bounding techniques

For the (complementary) first order loss function ($\widehat{\mathcal{L}}_{lb}(x,\omega)$) $\mathcal{L}_{lb}(x,\omega)$ the lower bound

$$\mathsf{E}[f(x,\omega)] \geq \sum_{i=1}^{N} p_i f(x, \mathsf{E}[\omega|\Omega_i])$$

is a piecewise linear function with $N+1$ segments.

Consider the bound presented above and let $f(x,\omega) = \max(x-\omega, 0)$,

$$\widehat{\mathcal{L}}_{lb}(x,\omega) = \sum_{i=1}^{N} p_i \max(x - \mathsf{E}[\omega|\Omega_i], 0)$$

this function is equivalent to

$$\widehat{\mathcal{L}}_{lb}(x,\omega) = \begin{cases} 0 & -\infty \leq x \leq \mathsf{E}[\omega|\Omega_1] \\ p_1 x - p_1 \mathsf{E}[\omega|\Omega_1] & \mathsf{E}[\omega|\Omega_1] \leq x \leq \mathsf{E}[\omega|\Omega_2] \\ (p_1+p_2)x - (p_1\mathsf{E}[\omega|\Omega_1] + p_2\mathsf{E}[\omega|\Omega_2]) & \mathsf{E}[\omega|\Omega_2] \leq x \leq \mathsf{E}[\omega|\Omega_3] \\ \vdots & \vdots \\ (p_1+p_2+\ldots+p_N)x - (p_1\mathsf{E}[\omega|\Omega_1] + \ldots + p_N\mathsf{E}[\omega|\Omega_N]) & \mathsf{E}[\omega|\Omega_{N-1}] \leq x \leq \mathsf{E}[\omega|\Omega_N] \end{cases}$$

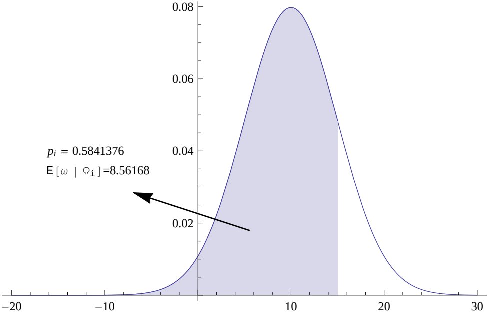
which is piecewise linear in x with breakpoints at $\mathsf{E}[\omega|\Omega_1], \mathsf{E}[\omega|\Omega_2], \ldots, \mathsf{E}[\omega|\Omega_N]$.

The first order loss function

Bounding techniques

$$p_i = \Pr\{\omega \in \Omega_i\} = \int_{\Omega_i} g_\omega(t)\, dt$$

$$\mathsf{E}[\omega|\Omega_i] = \frac{1}{p_i} \int_{\Omega_i} t g_\omega(t)\, dt$$



The first order loss function

Bounding techniques

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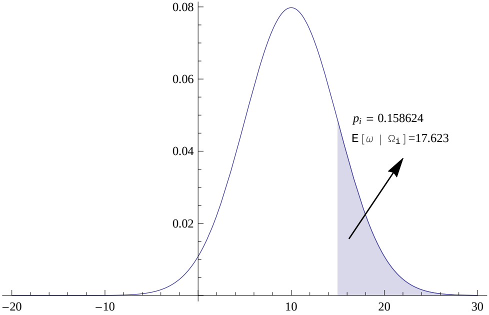

The first order loss function

Bounding techniques

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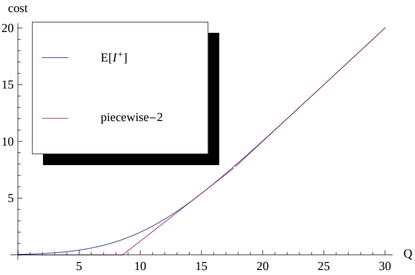

The first order loss function

Bounding techniques

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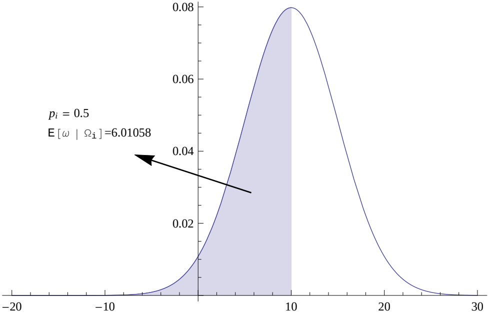

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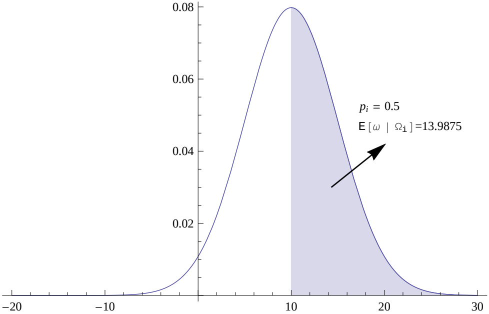

The first order loss function

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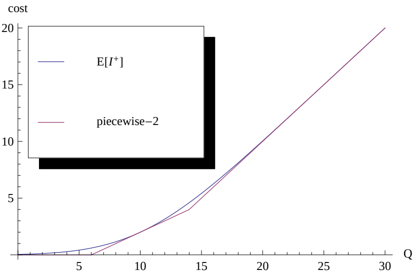


Static-dynamic uncertainty

MILP model under penalty cost

We introduce: $\tilde{I}_t^{lb}$ and $\tilde{I}_t^{ub}$ for $t = 1, \ldots, N$, which represent, respectively, a lower and an upper bound to the true value of $\mathrm{E}[\max(I_t, 0)]$.

We introduce: $\tilde{B}_t^{lb}$ and $\tilde{B}_t^{ub}$ for $t = 1, \ldots, N$, which represent a lower and upper bound, respectively, for the true value of $\mathrm{E}[-\min(I_t, 0)]$.

$$\mathrm{E}[\mathrm{TC}] = -vI_0 + v\sum_{t=1}^{N}\tilde{d}_t + \min\sum_{t=1}^{N}(K\delta_t + h\tilde{I}_t^{lb} + p\tilde{B}_t^{lb}) + v\tilde{I}_N$$

subject to, for $t = 1, \ldots N$

$$\tilde{I}_t + \tilde{d}_t - \tilde{I}_{t-1} \geq 0$$

$$\tilde{I}_t + \tilde{d}_t - \tilde{I}_{t-1} \leq \delta_t M_t$$

$$\sum_{j=1}^{t} P_{jt} = 1$$

$$P_{jt} \geq \delta_j - \sum_{k=j+1}^{t}\delta_k \qquad j = 1, \ldots, t$$

$$P_{jt} \in \{0, 1\} \qquad j = 1, \ldots, t$$

$$\delta_t \in \{0, 1\}$$

Static-dynamic uncertainty

MILP model under penalty cost

We introduce the following constraints in the model

$$\tilde{I}_t^{lb} \geq \tilde{I}_t \sum_{k=1}^{i} p_k - \sum_{j=1}^{t} \left(\sum_{k=1}^{i} p_k \mathsf{E}\{Z|\Omega_i\} \right) P_{jt} \sigma_{d_{j\ldots t}} \quad t = 1, \ldots, N; \; i = 1, \ldots, W$$

where $\sigma_{d_{j\ldots t}}$ denotes the standard deviation of $d_j + \ldots + d_t$ and $\tilde{I}_t^{lb} \geq 0$.

$$\tilde{I}_t^{ub} \geq \tilde{I}_t \sum_{k=1}^{i} p_k - \sum_{j=1}^{t} \left(\sum_{k=1}^{i} p_k \mathsf{E}\{Z|\Omega_i\} \right) P_{jt} \sigma_{d_{j\ldots t}} + \sum_{j=1}^{t} e^W P_{jt} \sigma_{d_{j\ldots t}} \quad \begin{array}{l} t = 1, \ldots, N, \\ i = 1, \ldots, W; \end{array}$$

where $\tilde{I}_t^{ub} \geq e^W$ and e^W denotes the maximum approximation error associated with a partition comprising W regions.

$$\tilde{B}_t^{lb} \geq -\tilde{I}_t + \tilde{I}_t \sum_{k=1}^{i} p_k - \sum_{j=1}^{t} \left(\sum_{k=1}^{i} p_k \mathsf{E}\{Z|\Omega_i\} \right) P_{jt} \sigma_{d_{j\ldots t}} \quad \begin{array}{l} t = 1, \ldots, N, \\ i = 1, \ldots, W; \end{array}$$

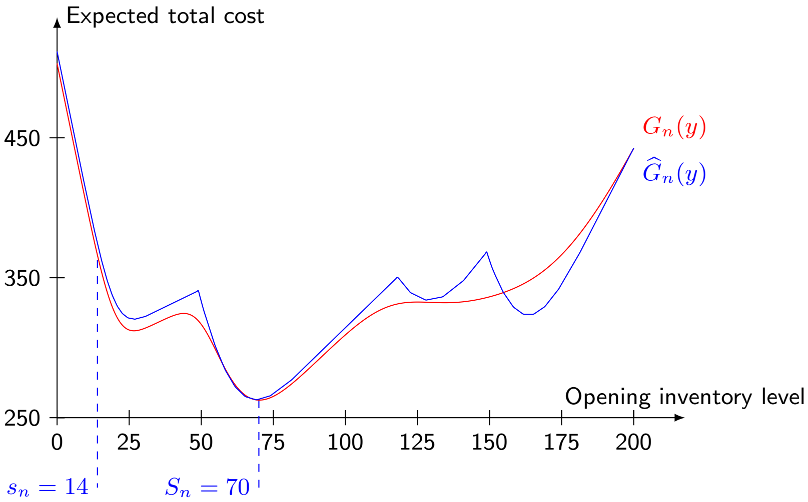
where $\tilde{B}_t^{ub} \geq -\tilde{I}_t$ and

$$\tilde{B}_t^{ub} \geq -\tilde{I}_t + \tilde{I}_t \sum_{k=1}^{i} p_k - \sum_{j=1}^{t} \left(\sum_{k=1}^{i} p_k \mathsf{E}\{Z|\Omega_i\} \right) P_{jt} \sigma_{d_{j\ldots t}} + \sum_{j=1}^{t} e^W P_{jt} \sigma_{d_{j\ldots t}} \quad \begin{array}{l} t = 1, \ldots, N, \\ i = 1, \ldots, W; \end{array}$$

where $\tilde{B}_t^{ub} \geq -\tilde{I}_t + e^W$.

Static-dynamic uncertainty

MILP model under penalty cost



MILP heuristic

Numerical example

Demand d_t normally distributed in each period t with mean $\mu_t \in \{20, 40, 60, 40\}$.

The standard deviation σ_t of the demand in period t is equal to $0.25\mu_t$.

Other problem parameters are $K = 100$, $h = 1$ and $p = 10$, and $v = 0$.

	SDP — ETC: (362.2,362.9)		MILP — ETC: (363.0,363.1)	
t	S_t	s_t	S_t	s_t
1	70.0	14.0	70.2	15.0
2	141	29.5	53.9	29.0
3	113	58.0	116	58.1
4	53.5	28.5	53.9	29.0

MILP heuristic

Preliminary computational results

K	5	10	20	50
v	0	0.5	1	2
p	2	5	10	15
c_v	0.1	0.2	0.3	

STA	10	10	10	10	10	10	10	10
LCY1	2.7	3.6	4.8	6.1	7.7	9.3	11	12.6
LCY2	15.3	16.2	16.6	16.6	16.2	15.3	14.1	12.6
SIN1	12.1	10	7.9	7	7.9	10	12.1	13
SIN2	15.7	10	4.3	2	4.3	10	15.7	18
RAND	20.9	9.1	3.3	7.9	0.2	7.6	10.9	11.5

Initial inventory set to zero, 1152 instances.

MILP heuristic

Preliminary computational results

Method	Journal	Avg. opt gap
Askin	AIIE Transactions 1978	2.09%
Bollapragada & Morton	Operations Research 1999	3.52%
New heuristic	Unpublished	0.2%

Conclusions

Related literature (sketch)

Edward Silver. Inventory control under a probabilistic time-varying, demand pattern.
A I I E Transactions, 10(4):371–379, December 1978

Ronald G. Askin. A procedure for production lot sizing with probabilistic dynamic demand.
A I I E Transactions, 13(2):132–137, June 1981

J. H. Bookbinder and J. Y. Tan. Strategies for the probabilistic lot-sizing problem with service-level constraints.
Management Science, 34:1096–1108, 1988

Charles R. Sox. Dynamic lot sizing with random demand and non-stationary costs.
Operations Research Letters, 20(4):155–164, May 1997

Conclusions

Related literature (sketch)

Srinivas Bollapragada and Thomas E. Morton. A simple heuristic for computing nonstationary (s, s) policies.
Oper. Res., 47(4):576–584, April 1999

S. Armagan Tarim and Brian G. Kingsman. Modelling and computing (Rn,Sn) policies for inventory systems with non-stationary stochastic demand.
European Journal of Operational Research, 174(1):581–599, October 2006

Roberto Rossi, S. Armagan Tarim, Steven Prestwich, and Brahim Hnich. Piecewise linear lower and upper bounds for the standard normal first order loss function.
Applied Mathematics and Computation, 231:489–502, March 2014

Roberto Rossi, Onur A. Kilic, and S. Armagan Tarim. Piecewise linear approximations for the static-dynamic uncertainty strategy in stochastic lot-sizing.
Omega, August 2014

Conclusions

Questions