Piecewise linear approximations of the standard normal first order loss function and an application to stochastic inventory control

Dr. Roberto Rossi

The University of Edinburgh Business School,
The University of Edinburgh, UK
roberto.rossi@ed.ac.uk

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This presentation illustrates results covered in the following working papers:

Roberto Rossi, S. Armagan Tarim, Brahim Hnich, and Steven D. Prestwich. Piecewise linear approximations of the standard normal first order loss function. Submitted to Applied Mathematics and Computation, arXiv:1307.1708, 2013

Roberto Rossi, Onur A. Kilic, and S. Armagan Tarim. Piecewise linear approximations for the static-dynamic uncertainty strategy in stochastic lot-sizing. Submitted to Omega, arXiv:1307.5942, 2013

Introduction

Related literature (sketch)

J. H. Bookbinder and J. Y. Tan. Strategies for the probabilistic lot-sizing problem with service-level constraints. *Management Science*, 34:1096–1108, 1988

S. A. Tarim and Brian G. Kingsman. The stochastic dynamic production/inventory lot-sizing problem with service-level constraints. *International Journal of Production Economics*, 88(1):105–119, March 2004

S. Armagan Tarim and Brian G. Kingsman. Modelling and computing (Rn,Sn) policies for inventory systems with non-stationary stochastic demand. *European Journal of Operational Research*, 174(1):581–599, October 2006

H. Tempelmeier. On the stochastic uncapacitated dynamic single-item lotsizing problem with service level constraints. *European Journal of Operational Research*, 181(1):184–194, August 2007

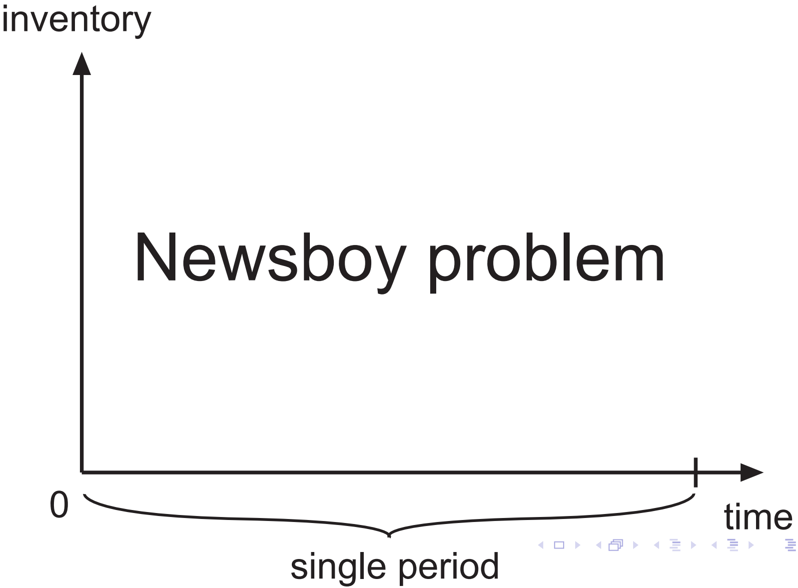
The investigation resorts to answering the following two key questions:

How can we produce "effective" piecewise linearisations of the first order loss function?

How can we employ these linearizations to model in a seamless way a number of variants of the stochastic lot-sizing problem under a static-dynamic uncertainty control policy, thus avoiding ad-hoc solutions?

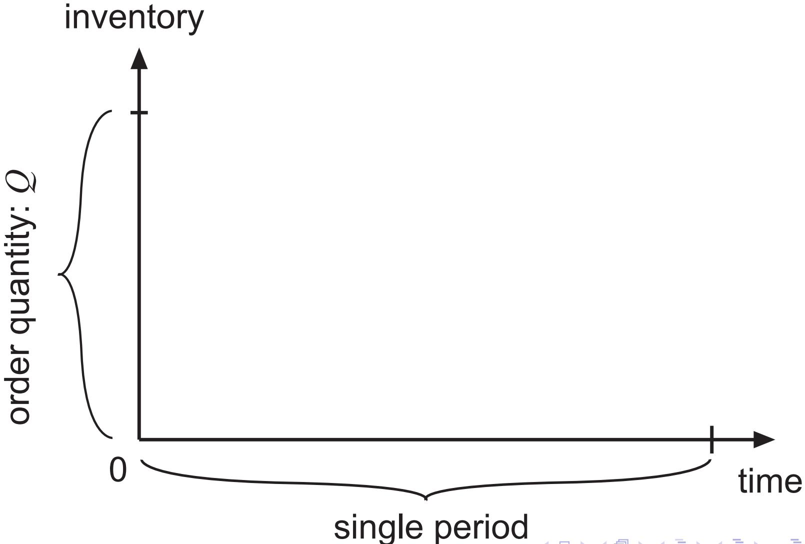
A motivating example

The newsboy problem



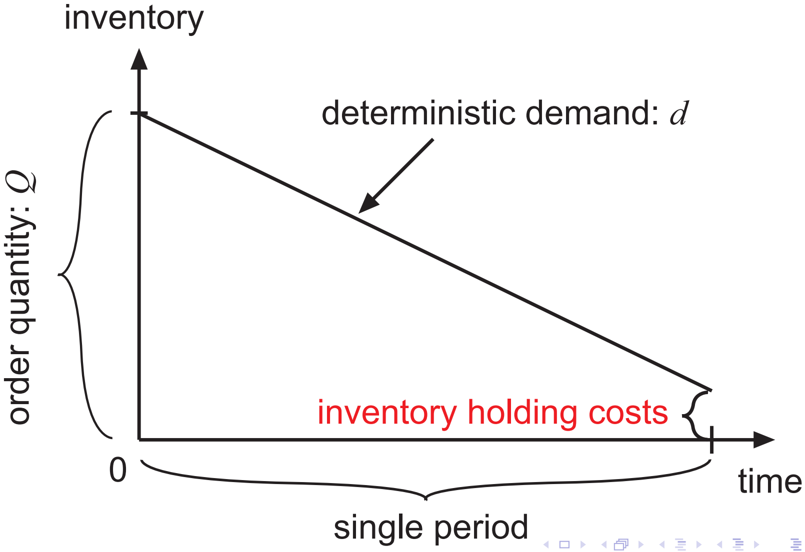
The newsboy problem

Order quantity



The newsboy problem

Deterministic demand



The newsboy problem

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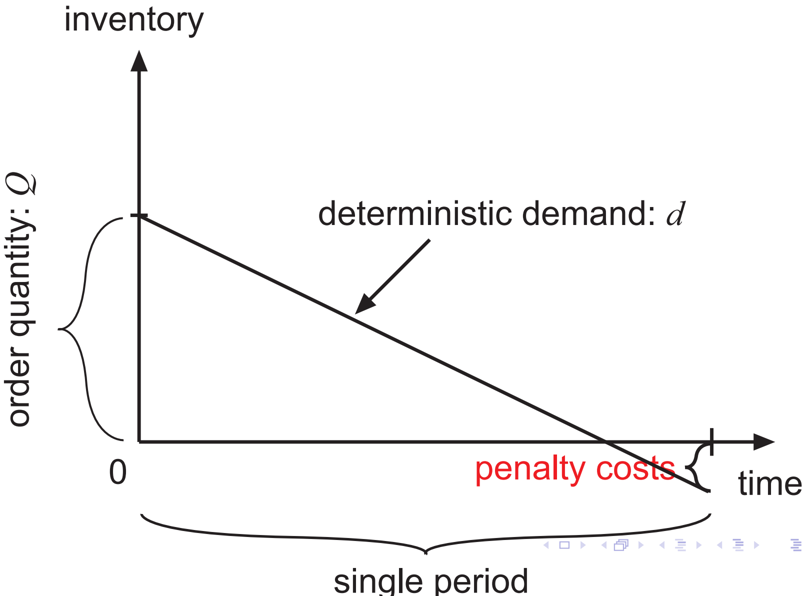

The newsboy problem

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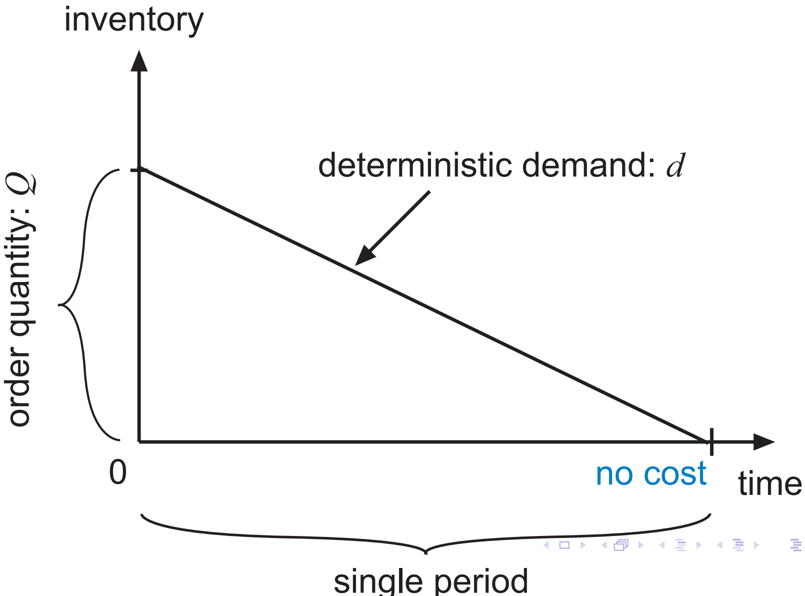
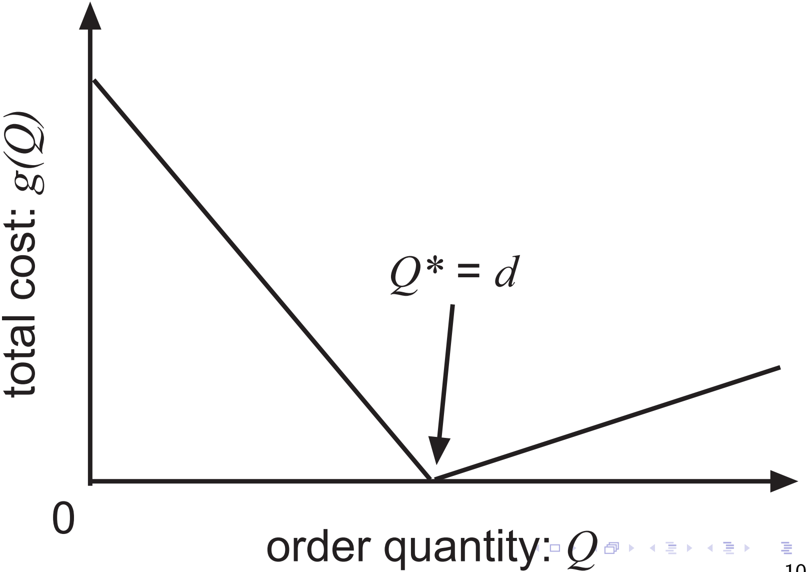


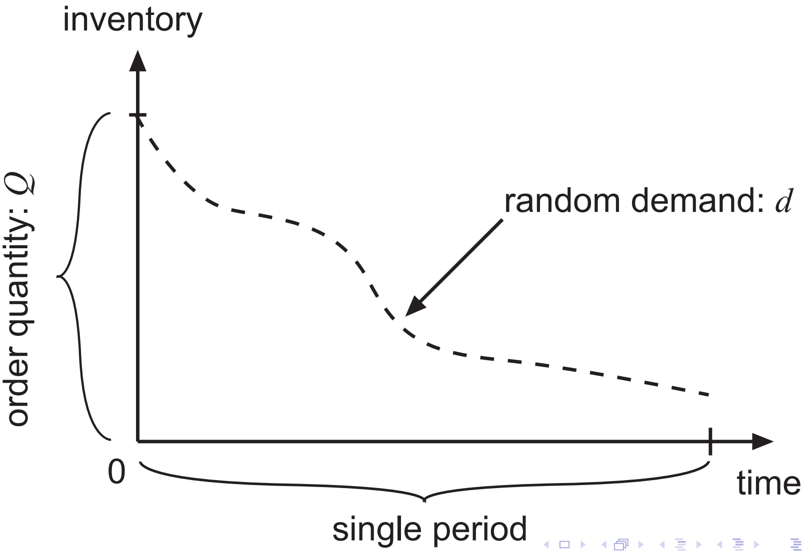
The newsboy problem

Cost structure under deterministic demand



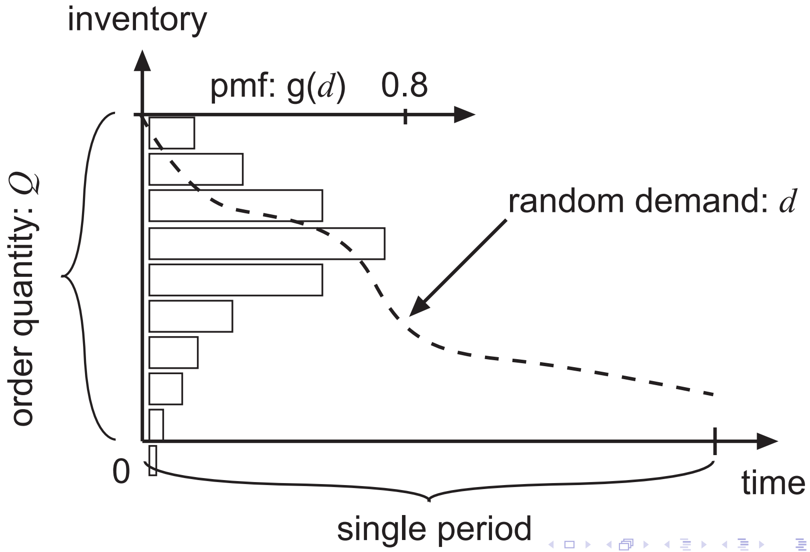
The newsboy problem

Random demand



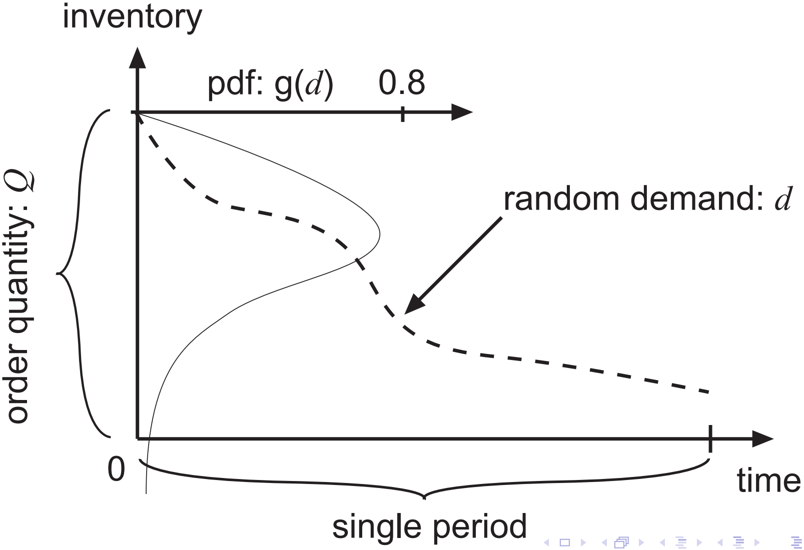
The newsboy problem

Random demand



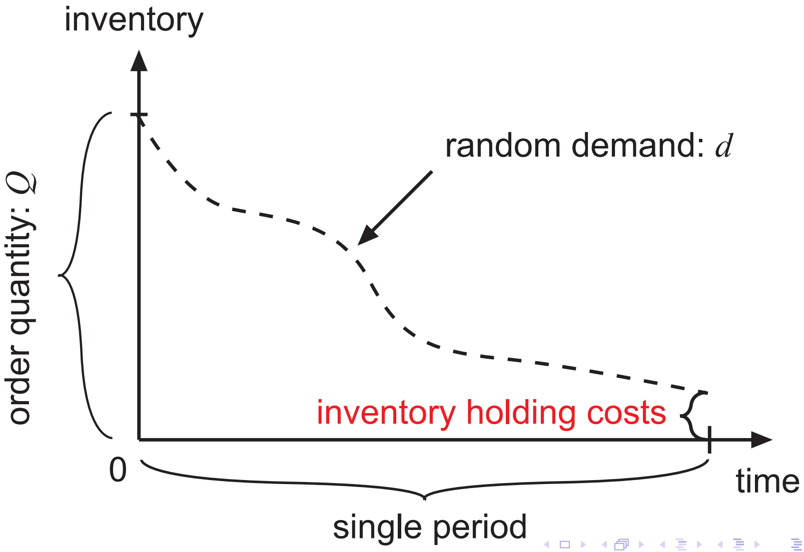
The newsboy problem

Random demand



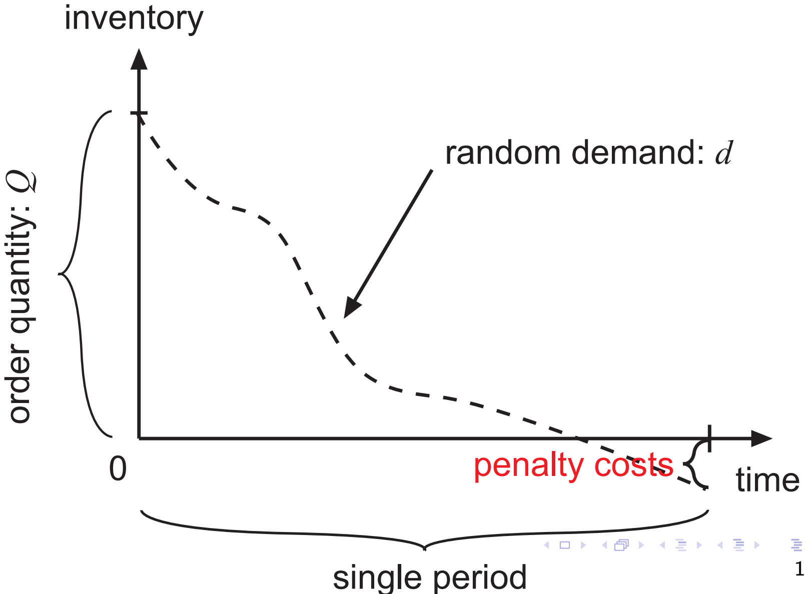
The newsboy problem

Cost structure under random demand



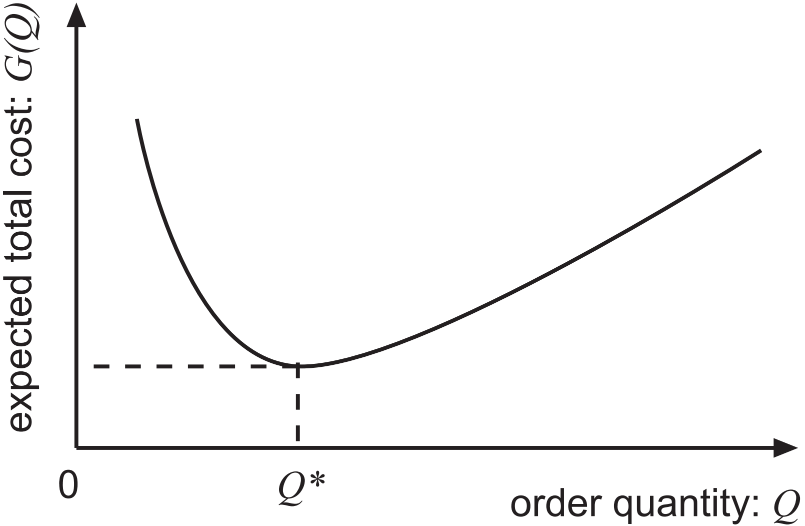
The newsboy problem

Cost structure under random demand



The newsboy problem

Cost structure under random demand



The newsboy problem

Mathematical formulation

Consider

- d: a **one-period** random demand that follows a **probability distribution** $f(d)$
- h: unit **holding cost**
- p: unit **penalty cost**

Let I be the end of period inventory and

$$g(I) = hI^+ + pI^-,$$

where $I^+ = \max(I, 0)$ and $I^- = -\min(I, 0)$.

The **expected total cost** is $G(Q) = E[g(Q - d)]$, where $E[\cdot]$ denotes the expected value.

The newsboy problem

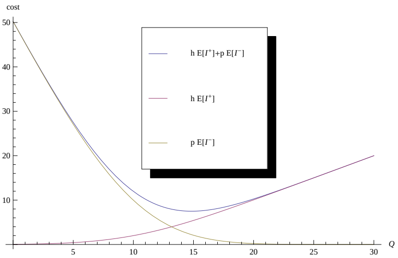
Mathematical formulation

Define:

$$E[I^+] = E[\max(Q-d,0)]: \quad \text{complementary first order loss function}$$
$$E[I^-] = E[\max(d-Q,0)]: \quad \text{first order loss function}$$

The **expected total cost** comprises two separable components

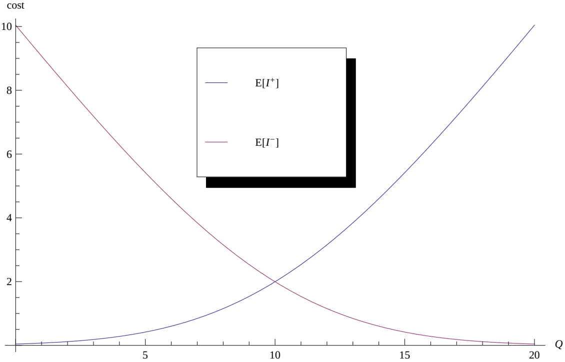
$$G(Q) = E[g(Q-d)] = hE[I^+] + pE[I^-]$$



$d = \text{Normal}(10, 5)$
$h =$ \$1
$p =$ \$5

The first order loss function

A graphical outlook



$E[I^+]$: complementary first order loss function
$E[I^-]$: first order loss function

The first order loss function

Properties

Consider a continuous random variable ω with support over $\mathbb{R}$, probability density function $g_\omega(x) : \mathbb{R} \to (0,1)$ and cumulative distribution function $G_\omega(x) : \mathbb{R} \to (0,1)$.

The first order loss function can be rewritten as

$$\mathcal{L}(x,\omega) = \int_{-\infty}^{\infty} \max(t-x,0) g_\omega(t)\, dt = \int_{x}^{\infty} (t-x) g_\omega(t)\, dt. \quad (1)$$

The complementary first order loss function can be rewritten as

$$\widehat{\mathcal{L}}(x,\omega) = \int_{-\infty}^{\infty} \max(x-t,0) g_\omega(t)\, dt = \int_{-\infty}^{x} (x-t) g_\omega(t)\, dt. \quad (2)$$

Lemma

$\mathcal{L}(x,\omega)$ and $\widehat{\mathcal{L}}(x,\omega)$ are convex in x.

The first order loss function

Properties

There is a close relationship between the first order loss function and the complementary first order loss function.

Lemma

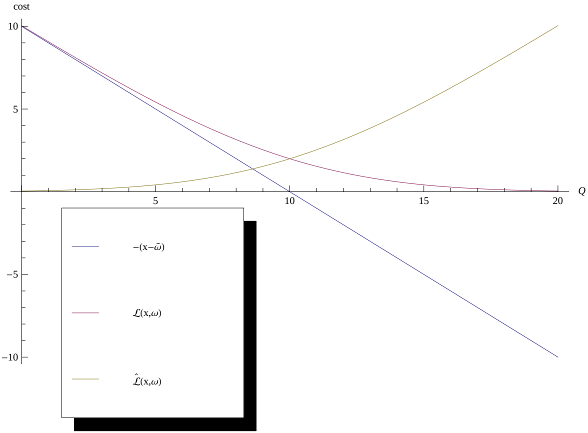
The first order loss function $\mathcal{L}(x,\omega)$ *can also be expressed as*

$$\mathcal{L}(x,\omega) = \widehat{\mathcal{L}}(x,\omega) - (x - \tilde{\omega}) \tag{3}$$

where $\tilde{\omega} = \mathsf{E}[\omega]$*.*

The first order loss function

Properties



The first order loss function

Properties

Lemma

The first order loss function $\mathcal{L}(x,\omega)$ can also be expressed as

$$\mathcal{L}(x,\omega) = \int_x^\infty (1 - G_\omega(t))\ dt \tag{4}$$

Lemma

The complementary first order loss function $\widehat{\mathcal{L}}(x,\omega)$ can also be expressed as

$$\widehat{\mathcal{L}}(x,\omega) = \int_{-\infty}^x G_\omega(t)\,dt. \tag{5}$$

These two results are not easily derived from each other!

The first order loss function

Properties for symmetric distributions

Lemma

If the probability density function of ω *is symmetric about a mean value* $\tilde{\omega}$*, then*

$$\mathcal{L}(x,\omega) = \widehat{\mathcal{L}}(2\tilde{\omega} - x, \omega).$$

Lemma

If the probability density function of ω *is symmetric about a mean value* $\tilde{\omega}$*, then*

$$\widehat{\mathcal{L}}(x,\omega) = \widehat{\mathcal{L}}(2\tilde{\omega} - x, \omega) + (x - \tilde{\omega})$$

and

$$\mathcal{L}(x,\omega) = \mathcal{L}(2\tilde{\omega} - x, \omega) - (x - \tilde{\omega}).$$

The first order loss function

Properties for normal distribution

Let ζ be a normally distributed random variable with mean μ and standard deviation σ.

Lemma

The complementary first order loss function of ζ can be expressed in terms of the standard Normal cumulative distribution function as

$$\widehat{\mathcal{L}}(x,\zeta) = \sigma \int_{-\infty}^{\frac{x-\mu}{\sigma}} \Phi(t)\,dt = \sigma \widehat{\mathcal{L}}\left(\frac{x-\mu}{\sigma}, Z\right), \tag{6}$$

where Z is a standard Normal random variable.

Unfortunately, no closed form expression exists for $\Phi(t)$.

The first order loss function

Non-linear approximations

Several approximations have been discussed for $\Phi(t)$, see e.g.

Marvin Zelen and Norman C. Severo. Probability functions. In Milton Abramowitz and Irene A Stegun, editors, *Handbook of Mathematical Functions*, volume 5 of *Applied Mathematics Series*, pages 925–995. GPO, 1964

Approximation to $\mathcal{L}(x, \zeta)$ have been recently discussed in

Steven K. De Schrijver, El-Houssaine Aghezzaf, and Hendrik Vanmaele. Double precision rational approximation algorithm for the inverse standard normal first order loss function. *Applied Mathematics and Computation*, 219(3):1375–1382, October 2012

The first order loss function

Non-linear approximations

Drawbacks

Existing approximations are non-linear and cannot be easily embedded in MILP models — ad-hoc strategies are needed.

Existing approximations do not provide upper and lower bounds for $\mathcal{L}(x, \zeta)$ — it is hard to estimate the goodness of the solutions obtained and to obtain optimality gaps.

The first order loss function

Piecewise linear approximations

We introduce a well-known inequality from stochastic programming

Peter Kall and Stein W. Wallace. *Stochastic Programming (Wiley Interscience Series in Systems and Optimization)*. John Wiley & Sons, August 1994, p. 167.

Theorem (Jensen's inequality)

Consider a random variable ω *with support* Ω *and a function* $f(x, s)$*, which for a fixed* x *is convex for all* $s \in \Omega$*, then*

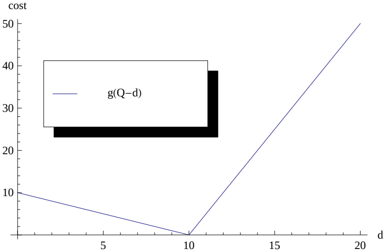
$$E[f(x, \omega)] \geq f(x, E[\omega]).$$

The first order loss function

The newsboy problem & Jensen's inequality

For a fixed Q, the **total cost** is convex for all values in the support of d.

$$g_Q(d) = g(Q - d) = h\max(Q - d, 0) + p\max(d - Q, 0))$$



$Q = 10$
$h =$ \$1
$p =$ \$5

The first order loss function

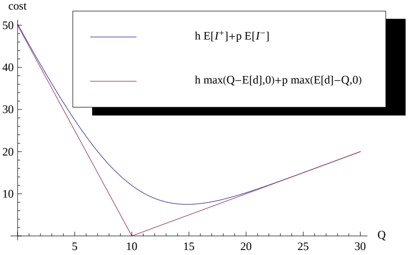
The newsboy problem & Jensen's inequality

Define:

$$\begin{aligned} E[I^+] &= E[\max(Q-d,0)]: \quad \text{complementary first order loss function} \\ E[I^-] &= E[\max(d-Q,0)]: \quad \text{first order loss function} \end{aligned}$$

The **expected total cost** can be bounded from below as follows.

$$hE[I^+]+pE[I^-] \geq h\max(Q-E[d],0)+p\max(E[d]-Q,0) = g(Q-\mathsf{E}[d])$$



$d = \text{Normal}(10, 5)$
$\mathsf{E}[d] = 10$
$h = \$1$
$p = \$5$

The first order loss function

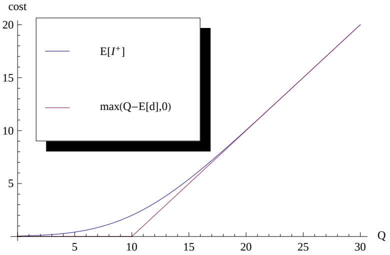
Bounding techniques

Define:

$$E[I^+] = E[\max(Q - d, 0)]: \quad \text{complementary first order loss function}$$

The complementary first order loss function can be bounded from below as follows.

$$E[I^+] \geq h \max(Q - E[d], 0)$$



$d =$ Normal$(10, 5)$
E$[d] = 10$
$h =$\$1
$p =$\$5

The first order loss function

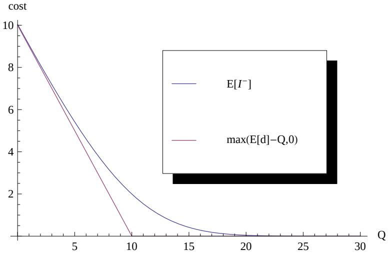
Bounding techniques

Define:

$$E[I^-] = E[\text{max}(d - Q, 0)]: \quad \text{first order loss function}$$

The first order loss function can be bounded from below as follows.

$$E[I^-] \geq \max(E[d] - Q, 0)$$



$d =$ Normal$(10, 5)$
E$[d] = 10$
$h =$ \$1
$p =$ \$5

The first order loss function

Bounding techniques

Let $g_\omega(\cdot)$ denote the probability density function of ω and consider a partition of the support Ω of ω into N disjoint compact subregions $\Omega_1, \ldots, \Omega_N$. We define, for all $i = 1, \ldots, N$

$$p_i = \Pr\{\omega \in \Omega_i\} = \int_{\Omega_i} g_\omega(t)\, dt$$

$$\mathsf{E}[\omega|\Omega_i] = \frac{1}{p_i} \int_{\Omega_i} t g_\omega(t)\, dt$$

Theorem

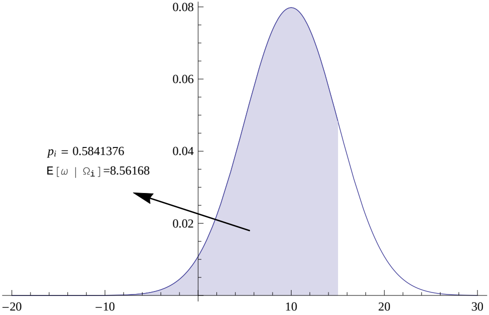
$$E[f(x,\omega)] \geq \sum_{i=1}^{N} p_i f(x, E[\omega|\Omega_i])$$

The first order loss function

Bounding techniques

$$p_i = \Pr\{\omega \in \Omega_i\} = \int_{\Omega_i} g_\omega(t)\,dt$$

$$\mathsf{E}[\omega|\Omega_i] = \frac{1}{p_i}\int_{\Omega_i} t g_\omega(t)\,dt$$



The first order loss function

Bounding techniques

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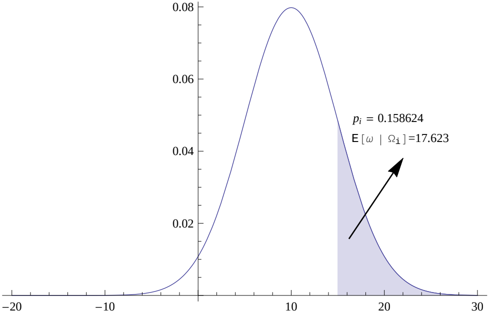


The first order loss function

Bounding techniques

For the (complementary) first order loss function ($\widehat{\mathcal{L}}_{lb}(x,\omega)$) $\mathcal{L}_{lb}(x,\omega)$ the lower bound

$$\mathsf{E}[f(x,\omega)] \geq \sum_{i=1}^{N} p_i f(x, \mathsf{E}[\omega|\Omega_i])$$

is a piecewise linear function with $N+1$ segments.

Consider the bound presented above and let $f(x,\omega) = \max(x-\omega, 0)$,

$$\widehat{\mathcal{L}}_{lb}(x,\omega) = \sum_{i=1}^{N} p_i \max(x - \mathsf{E}[\omega|\Omega_i], 0)$$

this function is equivalent to

$$\widehat{\mathcal{L}}_{lb}(x,\omega) = \begin{cases} 0 & -\infty \leq x \leq \mathsf{E}[\omega|\Omega_1] \\ p_1 x - p_1 \mathsf{E}[\omega|\Omega_1] & \mathsf{E}[\omega|\Omega_1] \leq x \leq \mathsf{E}[\omega|\Omega_2] \\ (p_1+p_2)x - (p_1\mathsf{E}[\omega|\Omega_1] + p_2\mathsf{E}[\omega|\Omega_2]) & \mathsf{E}[\omega|\Omega_2] \leq x \leq \mathsf{E}[\omega|\Omega_3] \\ \vdots & \vdots \\ (p_1+p_2+\ldots+p_N)x - (p_1\mathsf{E}[\omega|\Omega_1] + \ldots + p_N\mathsf{E}[\omega|\Omega_N]) & \mathsf{E}[\omega|\Omega_{N-1}] \leq x \leq \mathsf{E}[\omega|\Omega_N] \end{cases}$$

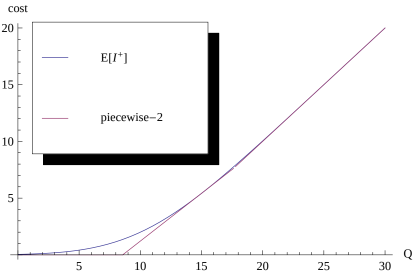
which is piecewise linear in x with breakpoints at $\mathsf{E}[\omega|\Omega_1], \mathsf{E}[\omega|\Omega_2], \ldots, \mathsf{E}[\omega|\Omega_N]$.

The first order loss function

Bounding techniques

$$p_i = \Pr\{\omega \in \Omega_i\} = \int_{\Omega_i} g_\omega(t)\,dt$$

$$\mathsf{E}[\omega|\Omega_i] = \frac{1}{p_i}\int_{\Omega_i} t g_\omega(t)\,dt$$



The first order loss function

Bounding techniques

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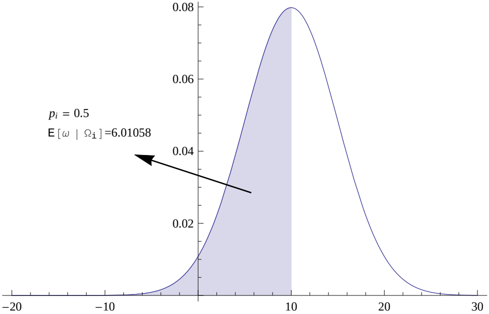

The first order loss function

Bounding techniques

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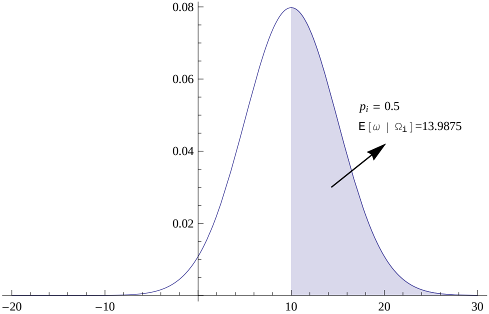

The first order loss function

Bounding techniques

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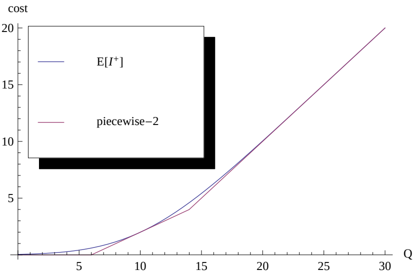


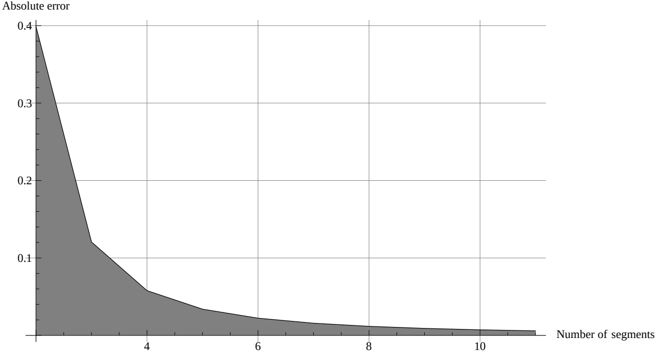
The first order loss function

Minimax optimal linearisation parameters for a standard normal random variable

Segments	Error		Piecewise linear approximation parameters									
		i	1	2	3	4	5	6	7	8	9	10
2	0.398942	b_i	∞									
		p_i	1									
		$E[\omega\|\Omega_i]$	0									
3	0.120656	b_i	0	∞								
		p_i	0.5	0.5								
		$E[\omega\|\Omega_i]$	−0.797885	0.797885								
4	0.0578441	b_i	−0.559725	0.559725	∞							
		p_i	0.287833	0.424333	0.287833							
		$E[\omega\|\Omega_i]$	−1.18505	0	1.18505							
5	0.0339052	b_i	−0.886942	0	0.886942	∞						
		p_i	0.187555	0.312445	0.312445	0.187555						
		$E[\omega\|\Omega_i]$	−1.43535	−0.415223	0.415223	1.43535						
6	0.0222709	b_i	−1.11507	−0.33895	0.33895	1.11507	∞					
		p_i	0.132411	0.234913	0.265353	0.234913	0.132411					
		$E[\omega\|\Omega_i]$	−1.61805	−0.691424	0	0.691424	1.61805					
7	0.0157461	b_i	−1.28855	−0.579834	0	0.579834	1.28855	∞				
		p_i	0.0987769	0.182236	0.218987	0.218987	0.182236	0.0987769				
		$E[\omega\|\Omega_i]$	−1.7608	−0.896011	−0.281889	0.281889	0.896011	1.7608				
8	0.0117218	b_i	−1.42763	−0.765185	−0.244223	0.244223	0.765185	1.42763	∞			
		p_i	0.0766989	0.145382	0.181448	0.192942	0.181448	0.145382	0.0766989			
		$E[\omega\|\Omega_i]$	−1.87735	−1.05723	−0.493405	0	0.493405	1.05723	1.87735			
9	0.00906529	b_i	−1.54317	−0.914924	−0.433939	0	0.433939	0.914924	1.54317	∞		
		p_i	0.0613946	0.118721	0.152051	0.167834	0.167834	0.152051	0.118721	0.0613946		
		$E[\omega\|\Omega_i]$	−1.97547	−1.18953	−0.661552	−0.213587	0.213587	0.661552	1.18953	1.97547		
10	0.00721992	b_i	−1.64166	−1.03998	−0.58826	−0.19112	0.19112	0.58826	1.03998	1.64166	∞	
		p_i	0.0503306	0.0988444	0.129004	0.146037	0.151568	0.146037	0.129004	0.0988444	0.0503306	
		$E[\omega\|\Omega_i]$	−2.05996	−1.30127	−0.8004	−0.384597	0.	0.384597	0.8004	1.30127	2.05996	
11	0.00588597	b_i	−1.72725	−1.14697	−0.717801	−0.347462	0.	0.347462	0.717801	1.14697	1.72725	∞
		p_i	0.0420611	0.0836356	0.110743	0.127682	0.135878	0.135878	0.127682	0.110743	0.0836356	0.0420611
		$E[\omega\|\Omega_i]$	−2.13399	−1.39768	−0.9182	−0.526575	−0.17199	0.17199	0.526575	0.9182	1.39768	2.13399

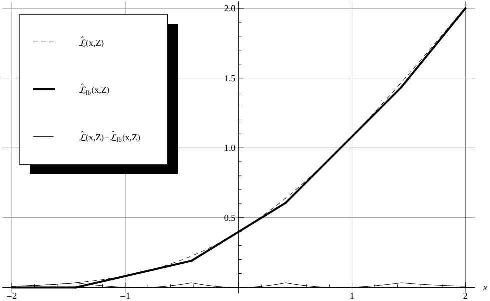
The first order loss function

Approximation error of $\widehat{\mathcal{L}}_{lb}(x, Z)$ with up to eleven segments



The first order loss function

Five-segment piecewise Jensen's bound for $\widehat{\mathcal{L}}(x,\zeta)$, where $\mu = 0$ and $\sigma = 1$



Five-segment piecewise Jensen's bound for $\widehat{\mathcal{L}}(x, Z)$, where Z is a standard normally distributed random variable. The maximum error is 0.0339052 and it is observed at $x \in \{\pm 1.43535, \pm 0.415223\}$.

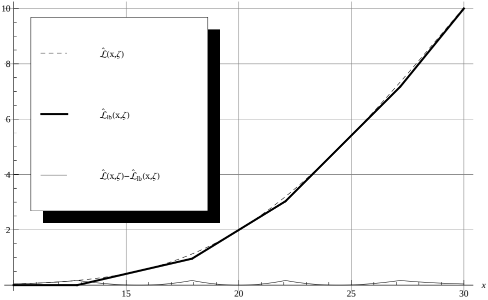
The first order loss function

Five-segment piecewise Jensen's bound for $\widehat{\mathcal{L}}(x,\zeta)$, where $\mu = 20$ and $\sigma = 5$

Below we exploit the fact that the complementary first order loss function of ζ can be expressed in terms of the standard Normal cumulative distribution function as

$$\widehat{\mathcal{L}}(x,\zeta) = \sigma \int_{-\infty}^{\frac{x-\mu}{\sigma}} \Phi(t)\,dt = \sigma \widehat{\mathcal{L}}\left(\frac{x-\mu}{\sigma}, Z\right),$$

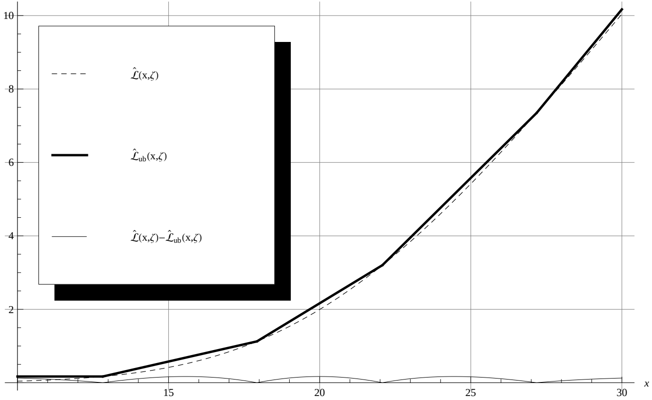
where Z is a standard Normal random variable.



Five-segment piecewise Jensen's bound for $\widehat{\mathcal{L}}(x,\zeta)$, where ζ is a normally distributed random variable with mean $\mu = 20$ and standard deviation $\sigma = 5$. The maximum error is $\sigma 0.0339052$ and it is observed at $x \in \{\sigma(\pm 1.43535) + \mu, \sigma(\pm 0.415223) + \mu\}$.

The first order loss function

Five-segment piecewise linear upper bound for $\widehat{\mathcal{L}}(x,\zeta)$, where $\mu = 20$ and $\sigma = 5$



Five-segment piecewise linear upper bound for $\widehat{\mathcal{L}}(x,\zeta)$, where ζ is a normally distributed random variable with mean $\mu = 20$ and standard deviation $\sigma = 5$. The maximum error is $\sigma 0.0339052$ and it is observed at $x \in \{\pm\infty, \sigma(\pm 0.886942) + \mu, \mu\}$.

Stochastic lot-sizing

General framework

$$\min \mathsf{E}[\mathsf{TC}] = \int_{d_1}\int_{d_2}\ldots\int_{d_N}\sum_{t=1}^{N}(a\delta_t + h\max(I_t,0) + vQ_t)\times$$
$$g_1(d_1)g_2(d_2)\ldots g_N(d_N)\,d(d_1)d(d_2)\ldots d(d_N)$$

subject to, for $t = 1,\ldots N$

$$I_t = I_0 + \sum_{i=1}^{t}(Q_i - d_i)$$

$$\delta_t = \begin{cases} 1 & \text{if } Q_t > 0, \\ 0 & \text{otherwise} \end{cases}$$

$$Q_i \geq 0,\ \ \delta_t \in \{0,1\}$$

Stochastic lot-sizing

α service level

$$\min \mathsf{E}[\mathsf{TC}] = \int_{d_1}\int_{d_2}\ldots\int_{d_N}\sum_{t=1}^{N}(a\delta_t + h\max(I_t,0) + vQ_t)\times$$
$$g_1(d_1)g_2(d_2)\ldots g_N(d_N)\,d(d_1)d(d_2)\ldots d(d_N)$$

subject to, for $t = 1,\ldots N$

$$I_t = I_0 + \sum_{i=1}^{t}(Q_i - d_i)$$

$$\delta_t = \begin{cases} 1 & \text{if } Q_t > 0, \\ 0 & \text{otherwise} \end{cases}$$

$$\Pr\{I_t \geq 0\} \geq \alpha$$

$$Q_i \geq 0,\ \delta_t \in \{0,1\}$$

Stochastic lot-sizing

Penalty cost

$$\min \mathsf{E}[\mathsf{TC}] = \int_{d_1}\int_{d_2}\ldots\int_{d_N}\sum_{t=1}^{N}$$

$$(a\delta_t + h\max(I_t,0) + p\max(-I_t,0) + vQ_t)\times$$

$$g_1(d_1)g_2(d_2)\ldots g_N(d_N)\,d(d_1)d(d_2)\ldots d(d_N)$$

subject to, for $t = 1,\ldots N$

$$I_t = I_0 + \sum_{i=1}^{t}(Q_i - d_i)$$

$$\delta_t = \begin{cases} 1 & \text{if } Q_t > 0, \\ 0 & \text{otherwise} \end{cases}$$

$$Q_i \geq 0,\ \ \delta_t \in \{0,1\}$$

Stochastic lot-sizing

β^{cyc} service level

H. Tempelmeier. On the stochastic uncapacitated dynamic single-item lotsizing problem with service level constraints. *European Journal of Operational Research*, 181(1):184–194, August 2007

$$\min \mathsf{E}[\mathsf{TC}] = \int_{d_1} \int_{d_2} \dots \int_{d_N} \sum_{t=1}^{N} (a\delta_t + h \max(I_t, 0) + vQ_t) \times$$
$$g_1(d_1) g_2(d_2) \dots g_N(d_N)\, d(d_1) d(d_2) \dots d(d_N)$$

subject to, for $t = 1, \dots N$

$$I_t = I_0 + \sum_{i=1}^{t} (Q_i - d_i)$$
$$\delta_t = \begin{cases} 1 & \text{if } Q_t > 0, \\ 0 & \text{otherwise} \end{cases}$$
$$1 - \max_{i=1,\dots,m} \left[\mathsf{E} \left\{ \frac{\text{Total backorders in replenishment cycle } i}{\text{Total demand in replenishment cycle } i} \right\} \right] \geq \beta^{\text{cyc}}$$
$$Q_i \geq 0,\ \ \delta_t \in \{0, 1\}$$

Stochastic lot-sizing

β service level

$$\min \mathsf{E}[\mathsf{TC}] = \int_{d_1}\int_{d_2}\ldots\int_{d_N}\sum_{t=1}^{N}(a\delta_t + h\max(I_t,0) + vQ_t)\times$$
$$g_1(d_1)g_2(d_2)\ldots g_N(d_N)\,d(d_1)d(d_2)\ldots d(d_N)$$

subject to, for $t = 1,\ldots N$

$$I_t = I_0 + \sum_{i=1}^{t}(Q_i - d_i)$$

$$\delta_t = \begin{cases} 1 & \text{if } Q_t > 0, \\ 0 & \text{otherwise} \end{cases}$$

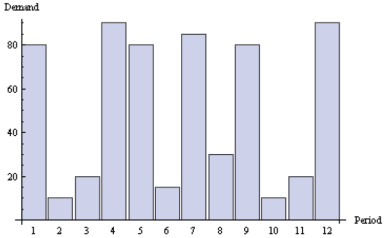
$$1 - \mathsf{E}\left\{\frac{\text{Total backorders within the planning horizon}}{\text{Total demand within the planning horizon}}\right\} \geq \beta$$

$$Q_i \geq 0,\;\; \delta_t \in \{0,1\}$$

Problem parameters

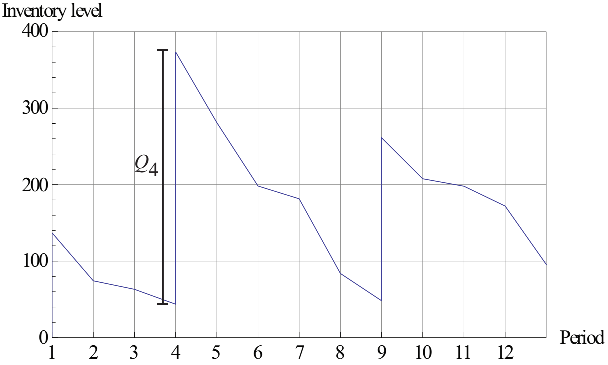
Normally distributed demand with constant coefficient of variation

$$c = \frac{\sigma_t}{\mu_t}$$



Static uncertainty

J. H. Bookbinder and J. Y. Tan. Strategies for the probabilistic lot-sizing problem with service-level constraints. *Management Science*, 34:1096–1108, 1988



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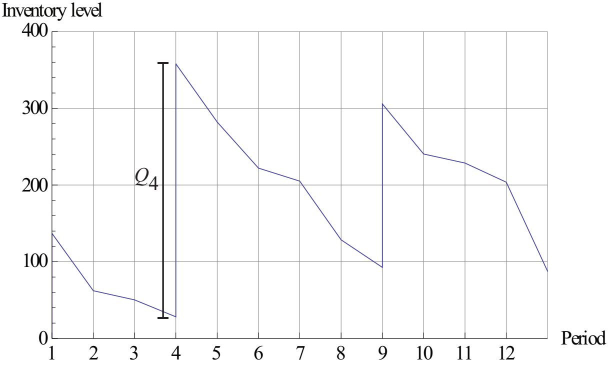

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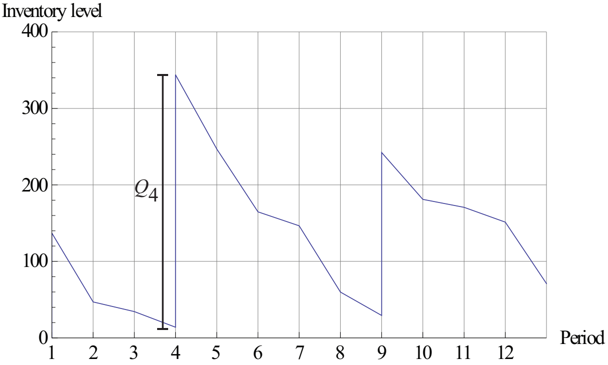

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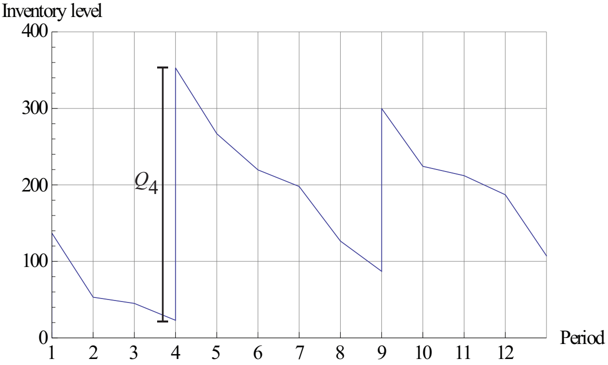

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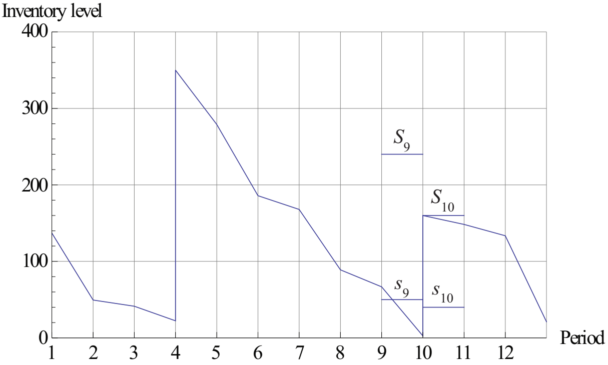

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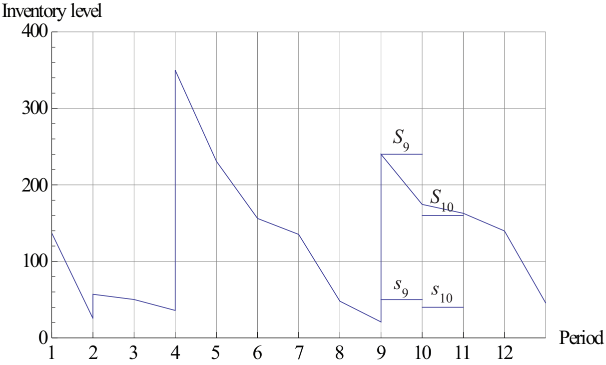

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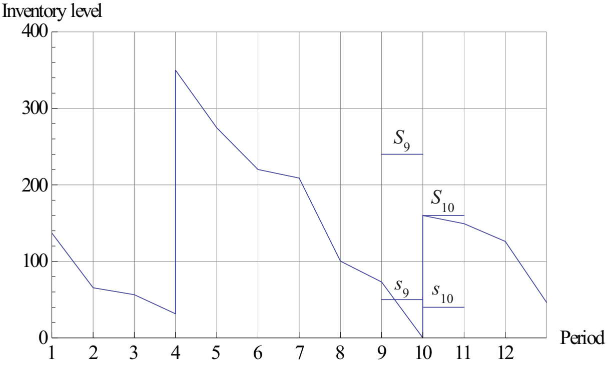

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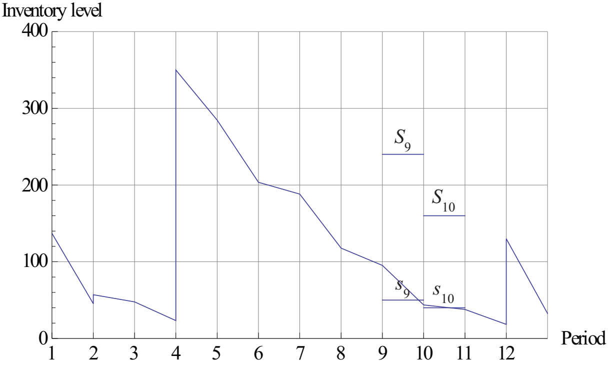

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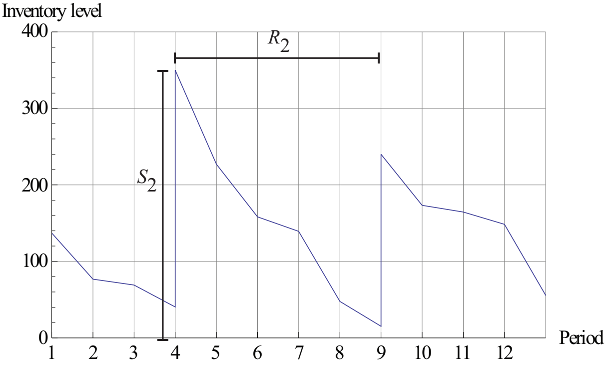

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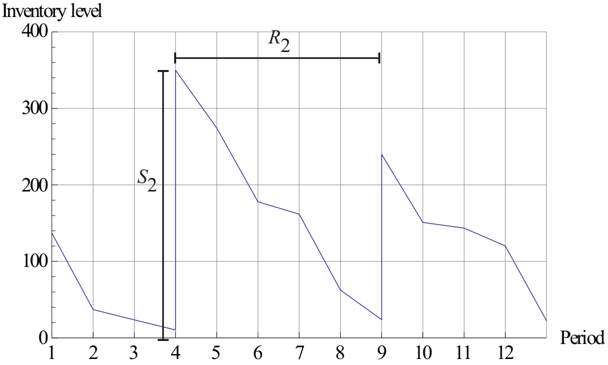

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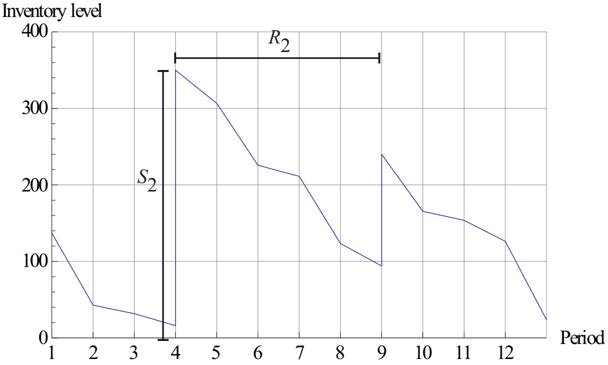

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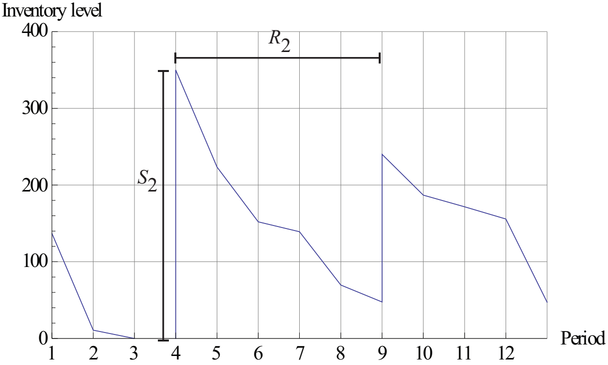


Static-dynamic uncertainty

MILP model under α service level

S. A. Tarim and Brian G. Kingsman. The stochastic dynamic production/inventory lot-sizing problem with service-level constraints. *International Journal of Production Economics*, 88(1):105–119, March 2004

$$\mathrm{E[TC]} = -vI_0 + v\sum_{t=1}^{N}\tilde{d}_t + \min\sum_{t=1}^{N}(a\delta_t + h\tilde{I}_t) + v\tilde{I}_N \tag{7}$$

subject to, for $t = 1, \dots N$

$$\tilde{I}_t + \tilde{d}_t - \tilde{I}_{t-1} \geq 0$$

$$\tilde{I}_t + \tilde{d}_t - \tilde{I}_{t-1} \leq \delta_t M_t$$

$$\tilde{I}_t \geq \sum_{j=1}^{t}\left(G^{-1}_{d_{j\dots t}}(\alpha) - \sum_{k=j}^{t}\tilde{d}_k\right)P_{jt}$$

$$\sum_{j=1}^{t} P_{jt} = 1$$

$$P_{jt} \geq \delta_j - \sum_{k=j+1}^{t}\delta_k \qquad j = 1, \dots, t$$

$$P_{jt} \in \{0, 1\} \qquad j = 1, \dots, t$$

$$\delta_t \in \{0, 1\}$$

Static-dynamic uncertainty

Enhanced MILP model under α service level

We introduce two new sets of decision variables: $\tilde{I}_t^{lb}$ and $\tilde{I}_t^{ub}$ for $t = 1, \ldots, N$. These represent, respectively, a lower and an upper bound to the true value of $\mathsf{E}[\max(I_t, 0)]$.

We introduce the following constraints in the model

$$\tilde{I}_t^{lb} \geq \tilde{I}_t \sum_{k=1}^{i} p_k - \sum_{j=1}^{t} \left(\sum_{k=1}^{i} p_k \mathsf{E}\{Z|\Omega_i\} \right) P_{jt} \sigma_{d_{j\ldots t}} \quad t = 1, \ldots, N; \; i = 1, \ldots, W$$

where $\sigma_{d_{j\ldots t}}$ denotes the standard deviation of $d_j + \ldots + d_t$ and $\tilde{I}_t^{lb} \geq 0$.

Consider a replenishment cycle covering periods $j, \ldots, t$ and associated order-up-to-level S. We aim to enforce $\tilde{I}_t^{lb} \geq \sigma \widehat{\mathcal{L}}_{lb}^{i} \left((S - \mu_{d_{j\ldots t}})/\sigma_{d_{j\ldots t}}, Z \right)$ for all $i = 1, \ldots, W$, where $\mu_{d_{j\ldots t}}$ is the expected value and $\sigma_{d_{j\ldots t}}$ the standard deviation of the demand over periods $j, \ldots, t$. Observe that $S - \mu_{d_{j\ldots t}} = \tilde{I}_t$, the above expression follows immediately.

$$\tilde{I}_t^{ub} \geq \tilde{I}_t \sum_{k=1}^{i} p_k - \sum_{j=1}^{t} \left(\sum_{k=1}^{i} p_k \mathsf{E}\{Z|\Omega_i\} \right) P_{jt} \sigma_{d_{j\ldots t}} + \sum_{j=1}^{t} e^W P_{jt} \sigma_{d_{j\ldots t}} \qquad \begin{array}{l} t = 1, \ldots, N, \\ i = 1, \ldots, W; \end{array}$$

where $\tilde{I}_t^{ub} \geq e^W$ and e^W denotes the maximum approximation error associated with a partition comprising W regions.

Static-dynamic uncertainty

Enhanced MILP model under α service level

Finally, the objective function then becomes

$$\mathsf{E}[\mathsf{TC}] = -vI_0 + v\sum_{t=1}^{N}\tilde{d}_t + \min\sum_{t=1}^{N}(a\delta_t + h\tilde{I}_t^{lb}) + v\tilde{I}_N \tag{8}$$

if our aim is to compute a lower bound for the cost of an optimal plan, or

$$\mathsf{E}[\mathsf{TC}] = -vI_0 + v\sum_{t=1}^{N}\tilde{d}_t + \min\sum_{t=1}^{N}(a\delta_t + h\tilde{I}_t^{ub}) + v\tilde{I}_N \tag{9}$$

if our aim is to compute an upper bound for the cost of an optimal plan.

Static-dynamic uncertainty

Numerical example under α service level

We demonstrate our approach on an instance originally discussed in S. A. Tarim and Brian G. Kingsman. The stochastic dynamic production/inventory lot-sizing problem with service-level constraints. *International Journal of Production Economics*, 88(1):105–119, March 2004. The instance comprises $N = 10$ periods in the planning horizon. Demand d_t in period t is normally distributed with mean μ_t and standard deviation σ_t.

t	1	2	3	4	5	6	7	8	9	10
μ_t	200	50	100	300	150	200	100	50	200	150
σ_t	60	15	30	90	45	60	30	15	60	45

Inventory holding costs are set to $h = 1$ setup costs are set to $a = 2500$; we target an α service level of 0.95. We ignore unit costs, i.e. $v = 0$.

Piecewise linear approximation (2 seg.) - E[TC]$\in$ $[9989.07, 10314.00]$

t	1	2	3	4	5	6	7	8	9	10
δ_t	1	0	0	0	0	1	0	0	0	0
S_t	1000.46	-	-	-	-	867.35	-	-	-	-

Piecewise linear approximation (11 seg.) - E[TC]$\in$ $[9993.66, 9998.46]$

t	1	2	3	4	5	6	7	8	9	10
δ_t	1	0	0	0	0	1	0	0	0	0
S_t	1000.46	-	-	-	-	867.35	-	-	-	-

The expected total cost estimated by Tarim and Kingsman's model is 9989.07. We simulated this policy and estimated its expected total cost with a margin of error of ± 0.001% at 95% confidence; the resulting cost is 9993.74 ± 0.1.

Static-dynamic uncertainty

MILP model under penalty cost

S. Armagan Tarim and Brian G. Kingsman. Modelling and computing (Rn,Sn) policies for inventory systems with non-stationary stochastic demand. *European Journal of Operational Research*, 174(1):581–599, October 2006

We introduce two new sets of variables $\tilde{B}_t^{lb}$ and $\tilde{B}_t^{ub}$ for $t = 1, \ldots, N$, which represent a lower and upper bound, respectively, for the true value of $\mathsf{E}[-\min(\tilde{I}_t, 0)]$ and thus allow us to compute lower and upper bounds for the expected backorders in each period.

$$\tilde{B}_t^{lb} \geq -\tilde{I}_t + \tilde{I}_t \sum_{k=1}^{i} p_k - \sum_{j=1}^{t} \left(\sum_{k=1}^{i} p_k \mathsf{E}\{Z|\Omega_i\} \right) P_{jt} \sigma_{d_{j \ldots t}} \qquad \begin{array}{l} t = 1, \ldots, N, \\ i = 1, \ldots, W; \end{array}$$

where $\tilde{B}_t^{ub} \geq -\tilde{I}_t$ and

$$\tilde{B}_t^{ub} \geq -\tilde{I}_t + \tilde{I}_t \sum_{k=1}^{i} p_k - \sum_{j=1}^{t} \left(\sum_{k=1}^{i} p_k \mathsf{E}\{Z|\Omega_i\} \right) P_{jt} \sigma_{d_{j \ldots t}} + \sum_{j=1}^{t} e^{W} P_{jt} \sigma_{d_{j \ldots t}} \qquad \begin{array}{l} t = 1, \ldots, N, \\ i = 1, \ldots, W; \end{array}$$

where $\tilde{B}_t^{ub} \geq -\tilde{I}_t + e^W$.

The objective function then becomes

$$\mathsf{E}[\mathsf{TC}] = -vI_0 + v \sum_{t=1}^{N} \tilde{d}_t + \min \sum_{t=1}^{N} (a\delta_t + h\tilde{I}_t^{lb} + b\tilde{B}_t^{lb}) + v\tilde{I}_N \qquad (10)$$

if our aim is to compute a lower bound for the cost of an optimal plan, or

$$\mathsf{E}[\mathsf{TC}] = -vI_0 + v \sum_{t=1}^{N} \tilde{d}_t + \min \sum_{t=1}^{N} (a\delta_t + h\tilde{I}_t^{ub} + b\tilde{B}_t^{ub}) + v\tilde{I}_N \qquad (11)$$

if our aim is to compute an upper bound for the cost of an optimal plan.

Static-dynamic uncertainty

Numerical example under penalty cost

We demonstrate our approach on an instance originally discussed in Charles R. Sox. Dynamic lot sizing with random demand and non-stationary costs. *Operations Research Letters*, 20(4):155–164, May 1997. The instance comprises $N = 8$ periods in the planning horizon. Demand d_t in period t is normally distributed with mean μ_t and standard deviation σ_t.

t	1	2	3	4	5	6	7	8
μ_t	110	40	10	62	12	80	122	130
σ_t	22	8	2	12.4	2.4	16	24.4	26
v_t	5.6	4.2	3.0	2.0	1.2	0.6	0.2	0

Piecewise linear approximation - E[TC]$\in [1024.70, 1034.24]$

t	1	2	3	4	5	6	7	8	9	10
δ_t	1	1	0	1	0	1	1	1		
S_t^{ub}	130.2	57.072	-	85.597	-	102.363	156.103	185.484		
S_t^{lb}	130.2	57.072	-	85.597	-	102.363	156.103	185.484		

Tarim and Kingsman - E[TC]=1031 (simulated: 1036.30)

t	1	2	3	4	5	6	7	8	9	10
δ_t	1	1	0	1	0	1	1	1		
S_t	128.5	56.9	-	84.6	-	101.9	155.4	165.6		

Inventory holding costs are set to $h = 0.5$; setup costs are set to $a = 48$; penalty costs are set to $b = 12$; finally, unit costs v_t vary from period to period. The initial inventory is set to 98 units.

The expected total cost estimated by Tarim and Kingsman's model is 1031. We simulated this policy and estimated its expected total cost with a margin of error of $\pm 0.01\%$ at 95% confidence; the resulting cost is 1036.30 ± 0.1.

Policy parameters obtained via our MILP approximation converge for eleven segments; the optimality gap is however 0.92%, reflecting the fact that the actual cost of this policy lies somewhere between 1024.70 and 1034.24. We simulated this policy and estimated its expected total cost with a margin of error of $\pm 0.01\%$ at 95% confidence; the resulting cost is 1034.14 ± 0.1.

Static-dynamic uncertainty

MILP model under β^{cyc} service level as defined in Tempelmeier (2007)

We introduce constraints

$$\tilde{B}_t^{lb} \leq (1 - \beta^{\text{cyc}}) \sum_{j=1}^{t} P_{jt} \mu_{d_{j \dots t}} \quad t = 1, \dots, N, \tag{12}$$

if our aim is to compute a lower bound for the cost of an optimal plan; or with

$$\tilde{B}_t^{ub} \leq (1 - \beta^{\text{cyc}}) \sum_{j=1}^{t} P_{jt} \mu_{d_{j \dots t}} \quad t = 1, \dots, N, \tag{13}$$

if our aim is to compute an upper bound for the cost of an optimal plan. Finally, the objective function becomes

$$\text{E[TC]} = -vI_0 + v \sum_{t=1}^{N} \tilde{d}_t + \min \sum_{t=1}^{N} (a\delta_t + h\tilde{I}_t^{lb}) + v\tilde{I}_N \tag{14}$$

if our aim is to compute a lower bound for the cost of an optimal plan, or

$$\text{E[TC]} = -vI_0 + v \sum_{t=1}^{N} \tilde{d}_t + \min \sum_{t=1}^{N} (a\delta_t + h\tilde{I}_t^{ub}) + v\tilde{I}_N \tag{15}$$

if our aim is to compute an upper bound for the cost of an optimal plan.

Static-dynamic uncertainty

Numerical example under β^{cyc} service level as defined in Tempelmeier (2007)

We solved the same instance discussed for the case of an α service level, however we now enforced a β^{cyc} service level of 0.95. By using eleven segments in the linearisation, the optimality gap is very narrow, i.e. 0.23%. We simulated both the policies obtained and estimated their expected total cost with a margin of error of $\pm 0.001\%$ at 95% confidence; the resulting costs are 8347.71 ± 0.08 and 8361.31 ± 0.08, respectively.

Piecewise linear approximation - E[TC]$\in [8347.40, 8367.03]$

t	1	2	3	4	5	6	7	8	9	10
δ_t	1	0	0	1	0	0	0	0	0	0
S_t^{ub}	373.95	-	-	1150.85	-	-	-	-	-	-
S_t^{lb}	372.84	-	-	1149.17	-	-	-	-	-	-

Tempelmeier - E[TC]=8348 (simulated: 8347.10)

t	1	2	3	4	5	6	7	8	9	10
δ_t	1	0	0	1	0	0	0	0	0	0
S_t	373	-	-	1149	-	-	-	-	-	-

It should be noted that an order-up-to-level of 1149, which is suggested in Tempelmeier's work for the second replenishment cycle, does not strictly meet the prescribed cycle service level, since it provides a cycle fill rate strictly lower than 0.95.

Static-dynamic uncertainty

Numerical example under β^{cyc} service level as defined in Tempelmeier (2007)

We consider the same instance, but now the cycle fill rate is set to $\beta^{\text{cyc}} = 0.6$ and the setup costs are reduced to $a = 1000$.

Piecewise linear approximation - E[TC]$\in [2773.63, 2781.10]$

t	1	2	3	4	5	6	7	8	9	10
δ_t	1	0	0	1	0	0	0	0	0	0
S_t^{ub}	210.71	-	-	694.84	-	-	-	-	-	-
S_t^{lb}	210.29	-	-	690.00	-	-	-	-	-	-

Tempelmeier - E[TC]=2776 (simulated: 2776.81)

t	1	2	3	4	5	6	7	8	9	10
δ_t	1	0	0	1	0	0	0	0	0	0
S_t	211	-	-	690	-	-	-	-	-	-

These results suggest that the model in Tempelmeier (2007) constitutes an excellent approach to the dynamic lot-sizing problem under non stationary stochastic demand and cycle fill rate constraints.

Static-dynamic uncertainty

MILP model under β service level

We introduce two new set of nonnegative variables $\tilde{C}_t^{lb}$ and $\tilde{C}_t^{ub}$ for $t = 0, \ldots, N$. These variables express the expected total backorders within the replenishment cycle that ends at period t, if there is one. Hence, $\tilde{C}_t^{lb}$ (resp. $\tilde{C}_t^{ub}$) should be equal to $\tilde{B}_t^{lb}$ (resp. $\tilde{B}_t^{ub}$), if t is the last period of a replenishment cycle; otherwise $\tilde{C}_t^{lb}$ (resp. $\tilde{C}_t^{ub}$) should be equal to 0. We enforce this fact as follows. For convenience, we set $\tilde{B}_0^{lb} = \tilde{B}_t^{ub} = \tilde{C}_0^{lb} = \tilde{C}_t^{ub} = I_0$, then we enforce

$$\tilde{C}_t^{lb} \geq \tilde{B}_t^{lb} - \delta_{t+1} \sum_{k=1}^{t} \tilde{d}_t \quad t = 0, \ldots, N-1, \tag{16}$$

$$\tilde{C}_t^{ub} \geq \tilde{B}_t^{ub} - \delta_{t+1} \sum_{k=1}^{t} \tilde{d}_t \quad t = 0, \ldots, N-1. \tag{17}$$

Finally, we must ensure that $\tilde{C}_N^{lb} = \tilde{B}_N^{lb}$ and $\tilde{C}_N^{ub} = \tilde{B}_N^{ub}$.

We then use these new variables to build constraint

$$\sum_{t=1}^{N} \tilde{C}_t^{lb} \leq (1-\beta) \sum_{t=1}^{N} \tilde{d}_t \tag{18}$$

which will replace (12), if our aim is to compute a lower bound for the cost of an optimal plan; and constraint

$$\sum_{t=1}^{N} \tilde{C}_t^{ub} \leq (1-\beta) \sum_{t=1}^{N} \tilde{d}_t \tag{19}$$

which will replace (13), if our aim is to compute an upper bound for the cost of an optimal plan.

Static-dynamic uncertainty

Numerical example under β service level

We solved the same instance discussed for the case of an α service level, however we now enforced a β service level of 0.95.

Piecewise linear approximation - E[TC]$\in [8313.48, 8335.38]$

t	1	2	3	4	5	6	7	8	9	10
δ_t	1	0	0	1	0	0	0	0	0	0
S_t^{ub}	413.12	-	-	1129.21	-	-	-	-	-	-
S_t^{lb}	413.12	-	-	1126.71	-	-	-	-	-	-

We simulated both the policies and estimated their expected total cost with a margin of error of $\pm 0.001\%$ at 95% confidence; the resulting costs are 8315.59 ± 0.08 and 8331.20 ± 0.08, respectively. This represents a 0.4% cost reduction with respect to the policies obtained via the model operating under a β^{cyc} service level.

Note that a control policy that orders up to 413.12 in period 1 and up to 1129.21 in period 4 is infeasible according to a cycle β service level constraint. The expected number of unit short in the second replenishment cycle amounts to 68.08 unit, that is 5.92% of the expected demand for this cycle, which amounts to 1150 units. However, the expected number of unit short over the planning horizon is 74.75 units, that is 6.67 units over the first replenishment cycle and 68.08 units over the second one. This represents 4.98% of the expected demand over the whole planning horizon, which amounts to 1500 units. Therefore this policy satisfies a classical β service level constraint.

Static-dynamic uncertainty

Numerical example under β service level

We consider the same instance, but now the cycle fill rate is set to $\beta^{\text{cyc}} = 0.6$ and the setup costs are reduced to $a = 1000$.

Piecewise linear approximation - E[TC]$\in [2602.58, 2612.69]$

t	1	2	3	4	5	6	7	8	9	10
δ_t	0	0	0	1	0	0	0	0	0	0
S_t^{ub}	-	-	-	903.49	-	-	-	-	-	-
S_t^{lb}	-	-	-	902.43	-	-	-	-	-	-

We simulated both the policies in Table 1 and estimated their expected total cost with a margin of error of $\pm 0.01\%$ at 95% confidence; the resulting costs are 2603.63 ± 0.26 and 2609.11 ± 0.26, respectively. For this instance, the cost reduction with respect to the policies obtained under a β^{cyc} service level is substantial and amounts to 6.4%.

Computational experience

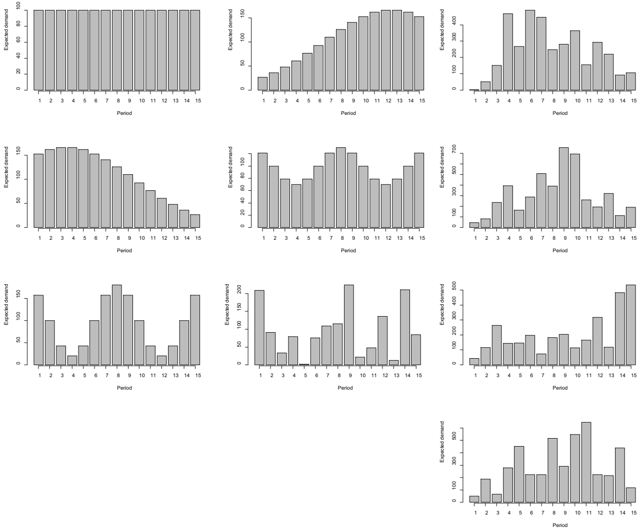
Instances

We generated a total of 810 instances.

10 demand patterns	
ordering cost	[500,1000,2000]
unit cost	[2,5,10]
coefficient of variation	[0.10,0.20,0.30]
penalty cost	[2,5,10]

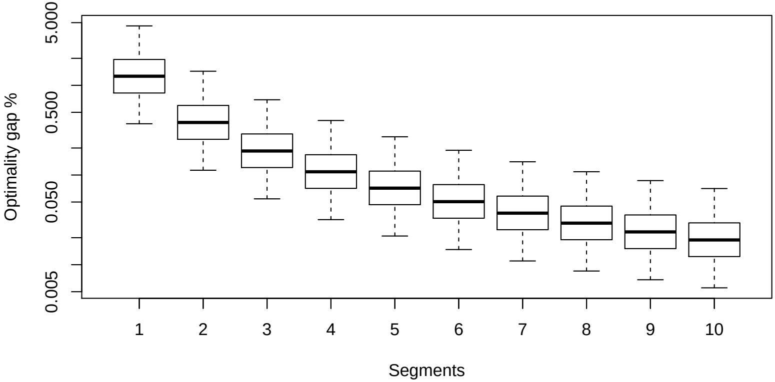
Computational experience

Demand patterns



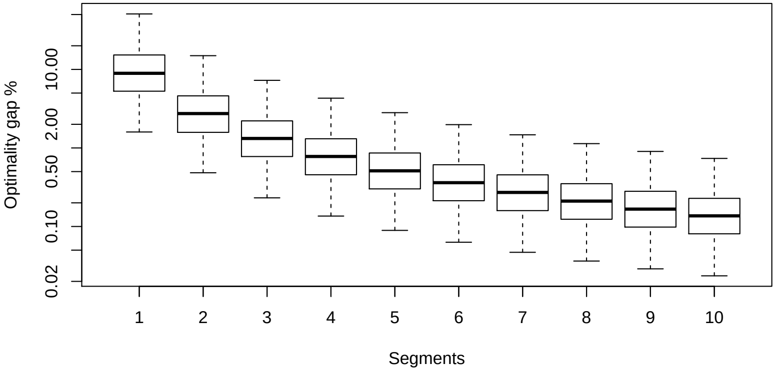
Computational experience

α service level



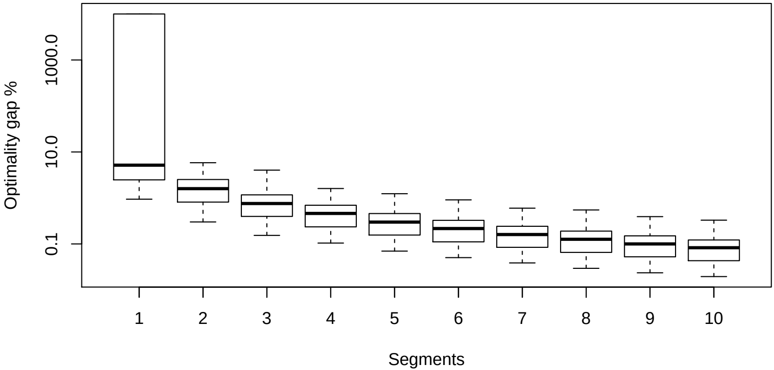
Computational experience

Penalty cost



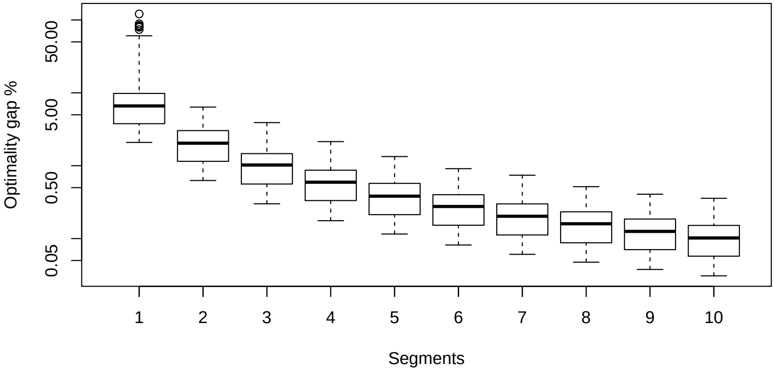
Computational experience

Cycle β service level



Computational experience

β service level



Conclusions

Questions