Dynamic Programming for Computing Non-Stationary (R,S) Policy Parameters

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   - A filtering procedure for the relaxed State Space Graph
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- Our approach significantly outperforms earlier works

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As the authors remark in [7], this definition of DP is somehow restrictive, but it will be sufficient for the discussion in this paper

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If the original problem is NP-hard, the correspondent state space graph will have an exponential number of nodes

State Space Relaxation (SSR)

Since the number of vertices in the state space graph is exponential, the concept of SSR has been widely studied in the Operations Research community, see (Christofides et al. [4])

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The relaxation represented by $RSG = (RS, RT)$ is, of course, useful if the number of vertices in the new graph is polynomial instead of exponential

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Property: An arc $a \in RT$ is necessarily preserved by the filtering procedure if there exists at least on arc $(s_i, s_j) \in T$ such that $(w(s_i), w(s_j)) = a$ and (s_i, s_j) is a candidate optimal arc in $SG = (S, T)$

Augmenting the Relaxed State Space Graph (Boland et al. [2])

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Property: A function $w^{\uparrow}(s)$ exists mapping every state $s \in S$, for which a state $w(s) \in RS$ exists, into a state $w^{\uparrow}(s) \in RS^{\uparrow}$, and every arc $(s_i, s_j) \in T$, for which an arc $(w(s_i), w(s_j)) \in RT$ exists, into an arc in $(w^{\uparrow}(s_i), w^{\uparrow}(s_j)) \in RT^{\uparrow}$

A Modified Dynamic Programming Algorithm with Filtering

Algorithm 1: `Relax - Filter - Augment`

input : a problem instance P
input : a relaxed state space graph $RSG(RS, RT)$ for P
input : a sound filtering algorithm $\mathcal{F}$ for the state space graph of P
input : an augmenting procedure $\mathcal{A}$ for the relaxed state space graph $RSG(RS, RT)$ of P
output: the optimal solution cost C

begin
  $RSG(\widehat{RS}, \widehat{RT}) \leftarrow RSG(RS, RT)$ filtered using $\mathcal{F}$;
  solve a shortest path problem (SPP) over $RSG(\widehat{RS}, \widehat{RT})$;
  if *the shortest path in* $RSG(\widehat{RS}, \widehat{RT})$ *produces a feasible solution for* P **then**
    $C \leftarrow$ cost of the shortest path in $RSG(\widehat{RS}, \widehat{RT})$;
  else
    $RSG(\widehat{RS}^{\uparrow}, \widehat{RT}^{\uparrow}) \leftarrow RSG(\widehat{RS}, \widehat{RT})$ augmented using $\mathcal{A}$;
    solve a shortest path problem (SPP) over $RSG(\widehat{RS}^{\uparrow}, \widehat{RT}^{\uparrow})$;
    $C \leftarrow$ cost of the shortest path in $RSG(\widehat{RS}^{\uparrow}, \widehat{RT}^{\uparrow})$;
end

Problem definition

INSTANCE Number $N \in \mathbb{Z}^+$ of periods, for each period i, $1 \leq i \leq N$, a random demand $d_i \in \mathbb{Z}_0^+$ with probability mass function $g_i(d_i)$. A production set-up cost $a \in \mathbb{Z}^+$, an inventory holding cost coefficient $h \in \mathbb{Z}^+$, a non-stock-out probability $0 < \alpha < 1$.
QUESTION Production amounts $X_i \in \mathbb{Z}_0^+$, and associated closing inventory levels $I_i = \sum_{t=1}^{i}(X_t - d_t)$, $1 \leq i \leq N$, such that $\Pr\{I_i \geq 0\} \geq \alpha$, and $E\left[\sum_{i=1}^{N}(h_i I_i) + \sum_{X_i>0} a_i\right]$, where $E[\cdot]$ is the expectation operator, is minimized?

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Assumptions

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Definition: A replenishment cycle, $T(i,j)$, is the time span between two consecutive orders occurring in periods i and $j+1$, $j \geq i$.

Definition: The cycle buffer stock, $b(i,j)$, denotes the minimum expected buffer stock level required to satisfy the required non-stock-out probability during $T(i,j)$.

Problem definition

The (R^n,S^n) policy

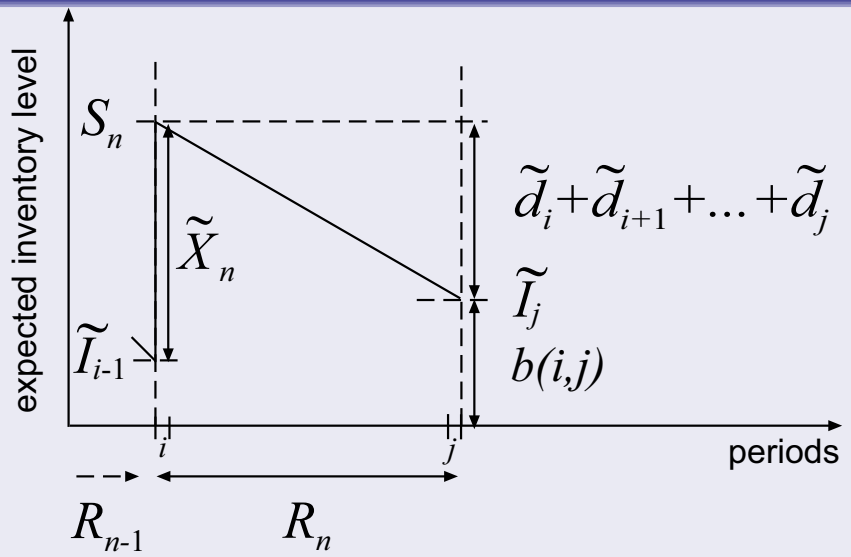

Figure: (R^n,S^n) policy. $\tilde{d}_i + \tilde{d}_{i+1} + \ldots + \tilde{d}_j$ is the expected demand over R^n; $b(i,j)$ is the minimum buffer stock required to guarantee service level α; $\tilde{X}_n$ is the expected order quantity in period i for replenishment cycle n; $\tilde{I}_{i-1}$ and $\tilde{I}_j$ are respectively the expected closing-inventory-levels for periods $i-1$ and j.

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Deterministic equivalent (Tarim & Kingsman [15])

INSTANCE Number $N \in \mathbb{Z}^+$ of periods, for each period i, $1 \leq i \leq N$, an expected demand $\tilde{d}_i \in \mathbb{Z}_0^+$. A production set-up cost $a \in \mathbb{Z}^+$, an inventory cost coefficient $h \in \mathbb{Z}^+$, and an $N \times N$ buffer stock matrix $b(\cdot, \cdot)$

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QUESTION Set of ordering decisions $\delta_i \in \{0, 1\}$, expected production amounts $\tilde{X}_i \in \mathbb{Z}_0^+$, and associated expected closing inventory levels $\tilde{I}_i = \sum_{t=1}^{i}(\tilde{X}_t - \tilde{d}_t)$, $1 \leq i \leq N$, such that $(\tilde{X}_i > 0) \Rightarrow (\delta_i = 1)$,

$$\tilde{I}_t \geq b(\max\{i | i \leq t \wedge \delta_i = 1\}, t),$$

and $\sum_{i=1}^{N} \left(a_i \delta_i + h_i \tilde{I}_i \right)$ is minimized.

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  - the inventory holding cost at the end of the period, which is equal to the (expected) closing-inventory-level s_k, multiplied by the per-unit holding cost h

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- X_k, the set of possible decisions at period k, comprises: the decision of not placing an order, the decision of covering 1 period with the order placed, the decision of covering 2 periods with the order placed, . . ., the decision of covering $N - k + 1$ periods with the order placed.

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- X_k, the set of possible decisions at period k, comprises: the decision of not placing an order, the decision of covering 1 period with the order placed, the decision of covering 2 periods with the order placed, ..., the decision of covering $N - k + 1$ periods with the order placed.
- S_k, the set of feasible (expected) closing-inventory-levels at the end of period k, is obtained recursively from the state transition functions $t_1, t_2, \ldots, t_k$, by assuming $s_0 = 0$ and that an order should be always placed at period 1

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$$f_i(s_i) = \min_{x_i \in X_i} \left\{ \min_{s_i \in S_{i-1}} \{ f_{i-1}(s_{i-1}) + p_i(s_i, x_i) \} \right\} \tag{3}$$

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where $s_i = t_i(s_{i-1}, x_i)$. In addition, we have the following boundary condition:

$f_1(s_1) = \min_{x_1 \in X_1} \{p_1(s_1, x_1)\}$, where $s_1 = t_1(s_0, x_1)$.

A State Space Relaxation (Tarim [13])

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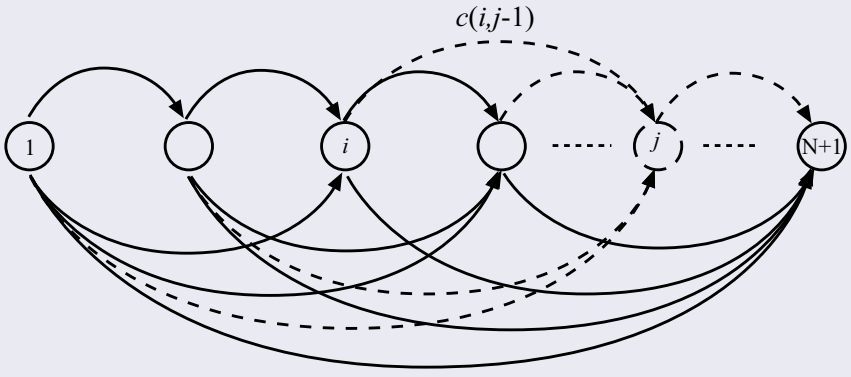

Figure: Shortest path problem graph

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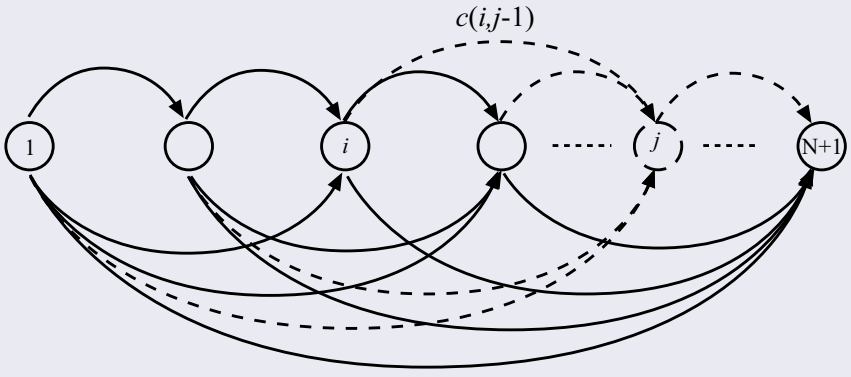

Figure: Shortest path problem graph

The cost of the shortest path from node 1 to node $N+1$ in the given graph is a valid lower bound for the original problem.

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- **Proposition (Tarim and Smith [16])** If $\forall k \in \{i,...,j-1\}$, $\big(c(i,k) + c(k+1,j) > c(i,j)\big) \vee \big(b(i,k) > R(k+1,j)\big)$ and $\exists k \in \{i,...,j\}$ $\big(c(i,k) + c(k+1,j+1) \leq c(i,j+1)\big) \wedge \big(b(i,k) \leq R(k+1,j+1)\big)$ then for period i the optimum length replenishment cycle is $T(i,p)^*$ where $i \leq p \leq j$, and j indicates an upper bound.

Filtering procedure for the relaxed State Space Graph (Tarim & Smith [16])

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- Since we have an upper bound j for the length of an optimum replenishment cycle starting at period i, we can remove from our graph every arc (i,t), where $t > j+1$.

Augmenting procedure for the relaxed State Space Graph

- This process is repeated for each node $i \in \{1, \ldots, N+1\}$, therefore we will only describe the steps performed on a single node

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- consider every inbound arc (p, i) at node i with associated buffer b^*
- for each outbound arc (i, j) in the graph we check that $b^* \leq R(i, j-1)$
- if this condition is satisfied for every outbound arc, then we preserve the inbound arc (p, i) at node i with the associated buffer b^*

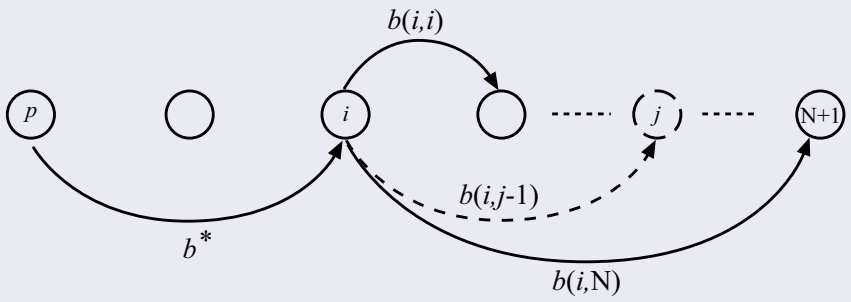

Figure: Feasible node point

Augmenting procedure for the relaxed State Space Graph

- If for some pair of subsequent replenishment cycles a negative order quantity is scheduled, in order to resolve this infeasibility we operate the following transformation:
- We introduce a new node i' in the graph
- We remove arc (p, i) and we introduce a new arc (p, i') with associated buffer b^*

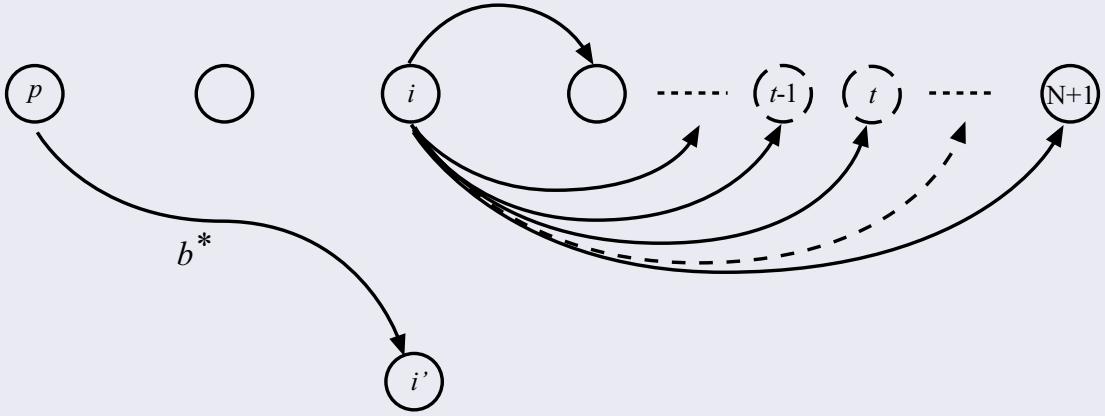

Figure: Infeasible node point

Augmenting procedure for the relaxed State Space Graph

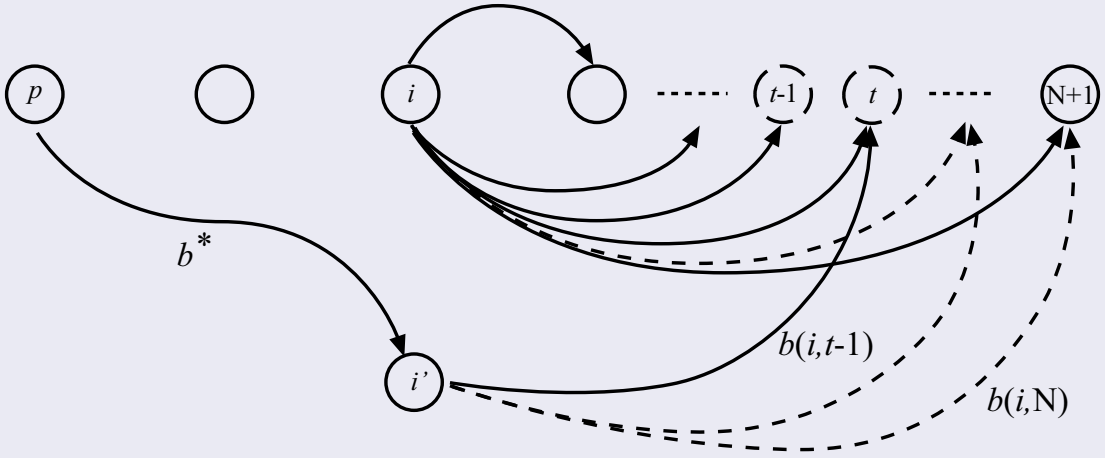

Figure: Infeasible node point

Then we connect this new node the following way:
Let $t > i$ be the minimum index for which $b^* \leq R(i, t-1) \leq R(i, t) \leq \ldots \leq R(i, N)$. We introduce arc (i', t) with buffer $b(i, t-1)$. Then, for each arc $(i, t+1), \ldots, (i, N+1)$ *in the graph*, we also introduce $(i', t+1), \ldots, (i', N+1)$ with buffer respectively $b(i, t), \ldots, b(i, N)$. It should be noted that some of the arcs $(i, t+1), \ldots, (i, N+1)$ may have been removed by the filtering

Augmenting procedure for the relaxed State Space Graph

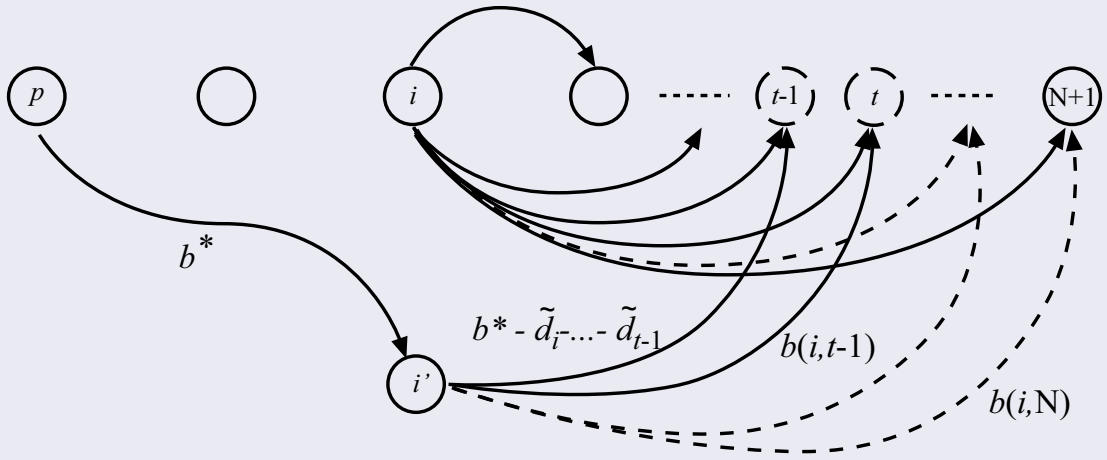

Figure: Infeasible node point

Then we connect this new node the following way:
Let $t-1 > i$ be the maximum index for which $b^* > \ldots \geq R(i, t-2)$. We introduce arc $(i', t-1)$ with buffer $b^* - \sum_{k=i}^{t-1} \tilde{d}_k$. Obviously arcs $(i', t-2), (i', t-3), \ldots$ are suboptimal and should not be introduced, since the inventory carried on from the former replenishment cycle is enough to cover subsequent periods up to $t-1$

Input data

- 5-period planning horizon

Input data

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- expected values for period demand are $[100, 125, 25, 40, 30]$

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- normally distributed demand for every period with a coefficient of variation $\sigma_t / \tilde{d}_t = 0.3$
- ordering cost value $a = 50$
- holding cost $h = 1$
- non-stock-out probability in each period set to $\alpha = 0.95$

Relax & Filter

Firstly we build the connection matrix for the relaxed problem...

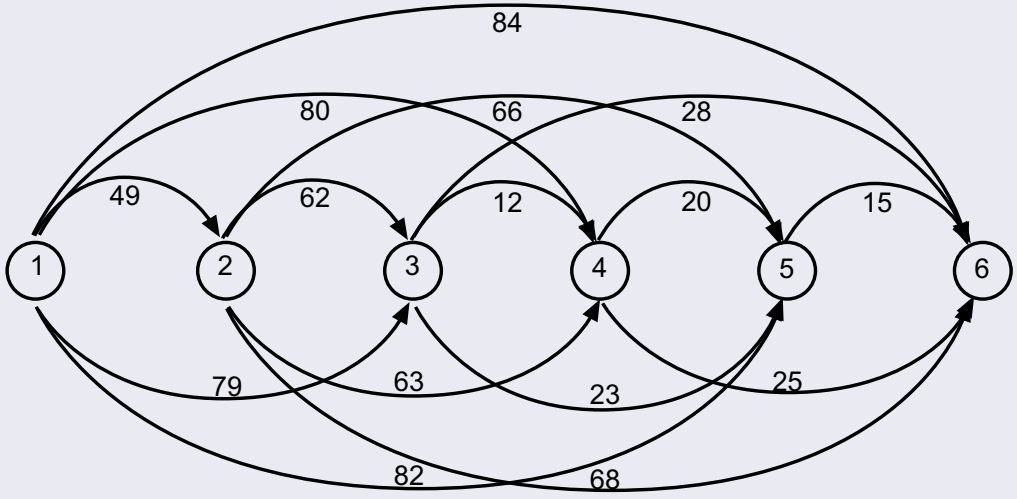


Figure: Connection matrix with expected buffer stock levels

Relax & Filter

Firstly we build the connection matrix for the relaxed problem...

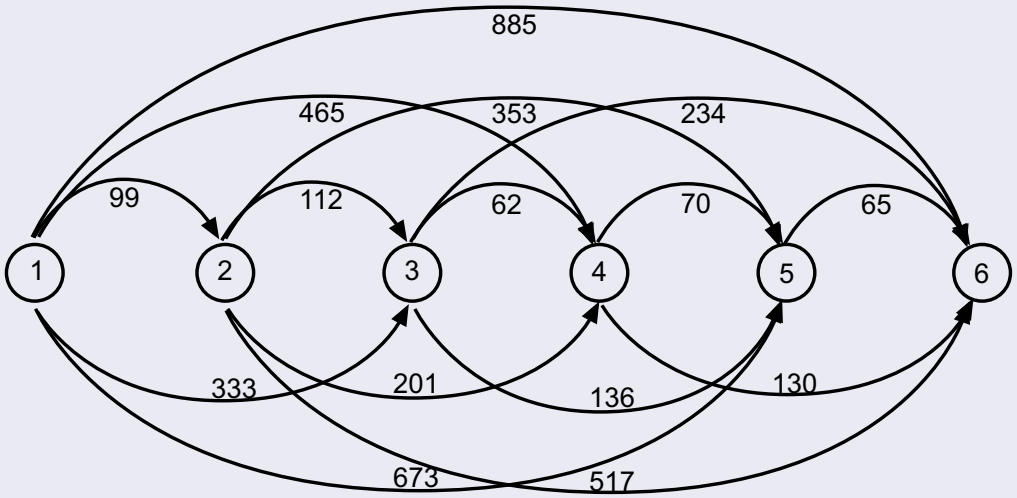


Figure: Connection matrix with expected cycle costs

Relax & Filter

Firstly we build the connection matrix for the relaxed problem...
...and we filter it by using the procedure already described...

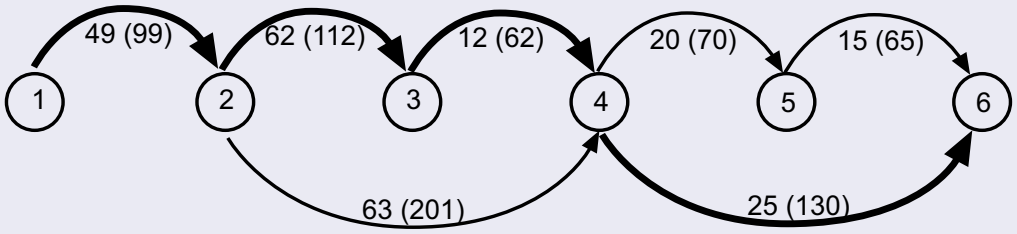


Figure: Filtered connection matrix. Expected buffer stock levels and expected cycle costs (in parenthesis) are shown for each arc. The shortest path is highlighted

The shortest path in this reduced network has a cost of 403. The order periods and the expected order quantities are respectively $[1, 2, 3, 4]$ and $[149, 138, -25, 83]$

Relax & Filter

Firstly we build the connection matrix for the relaxed problem...
...obtaining the following plan...

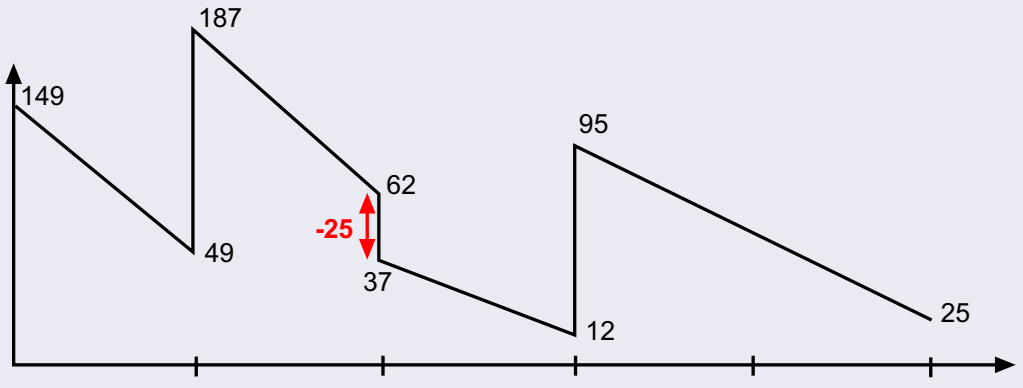
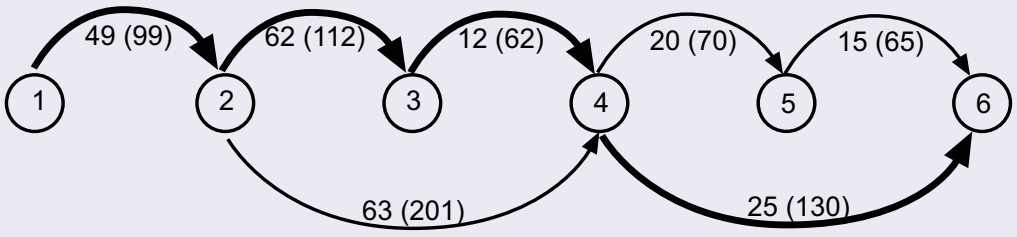


Figure: The respective plan

Augment



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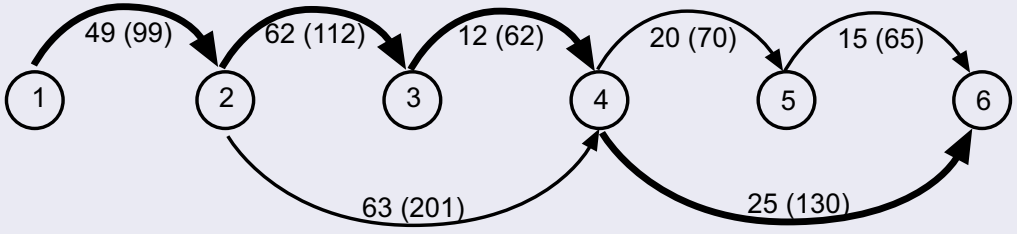
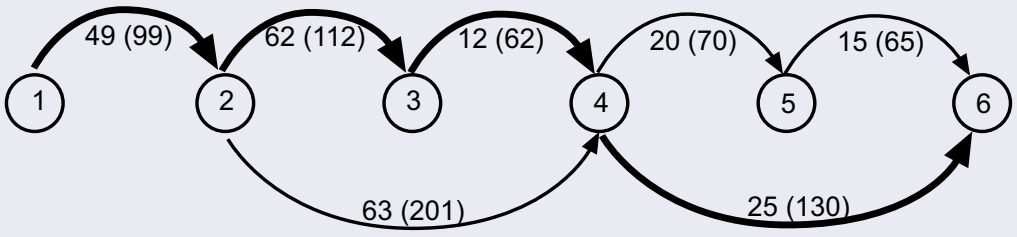


According to the procedure described we augment the filtered graph...

Augment



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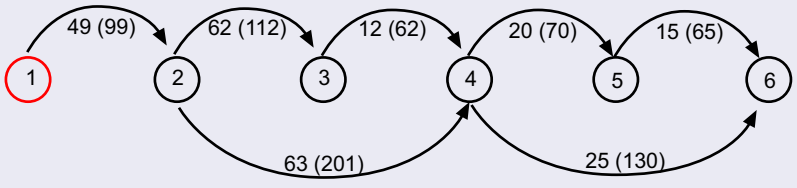

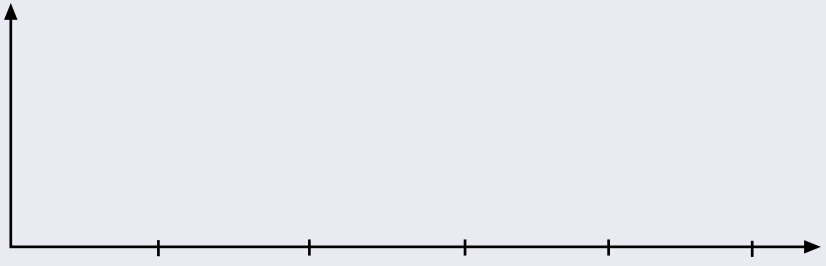

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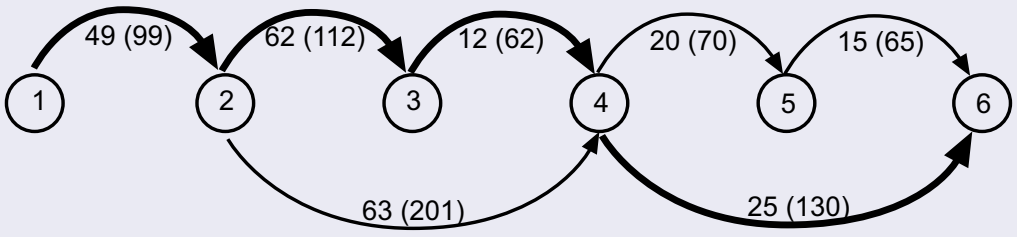

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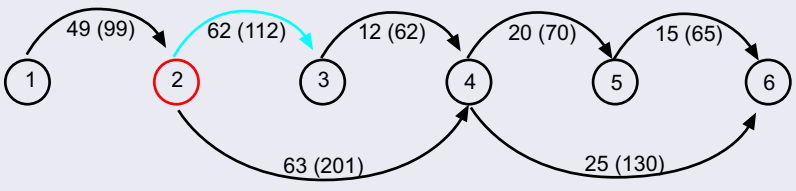

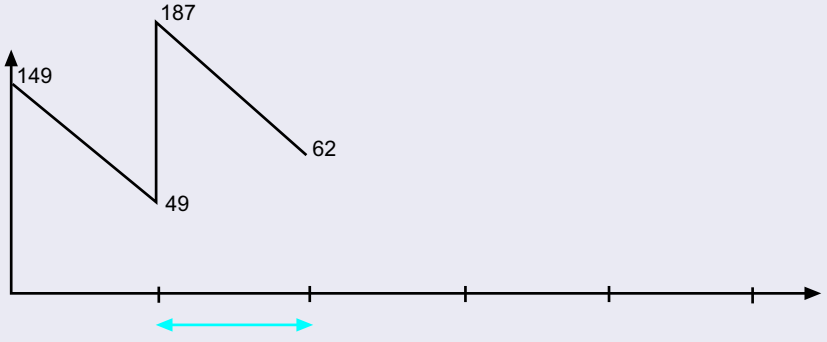

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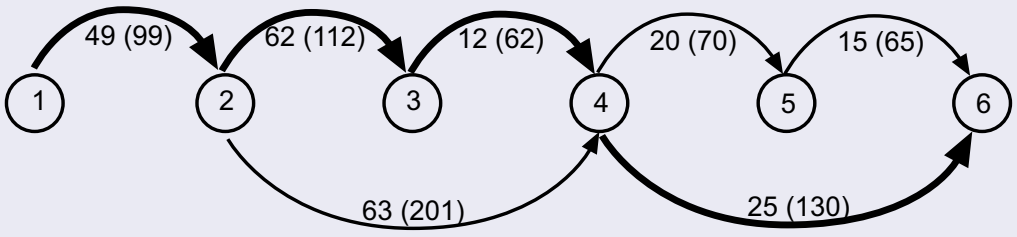

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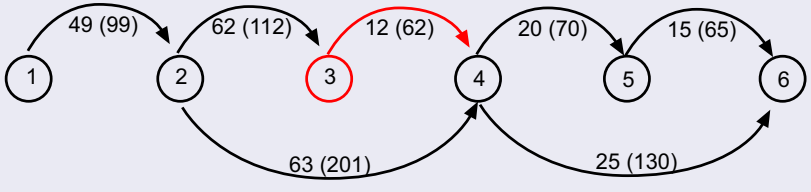

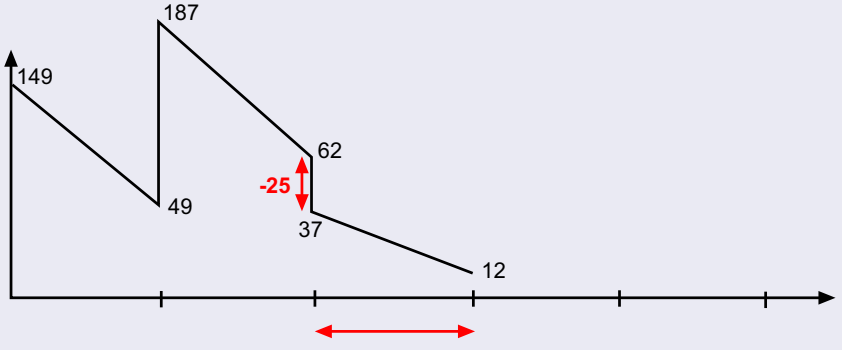

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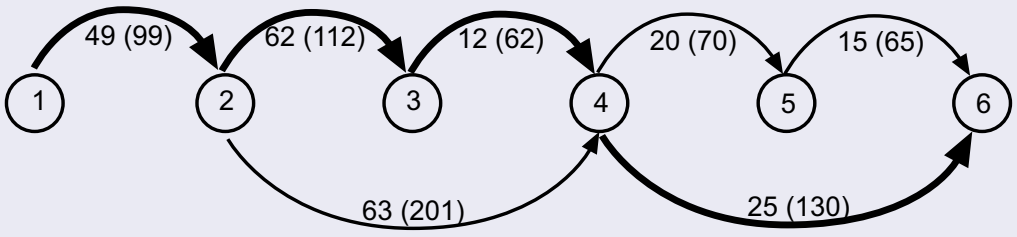

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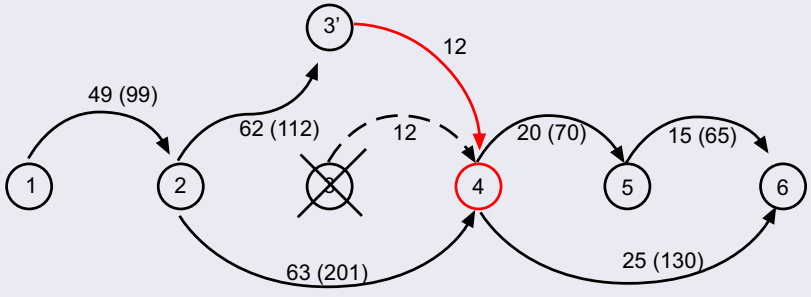

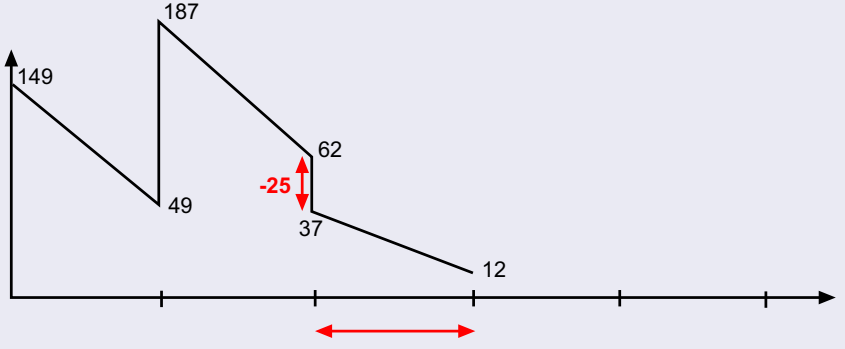

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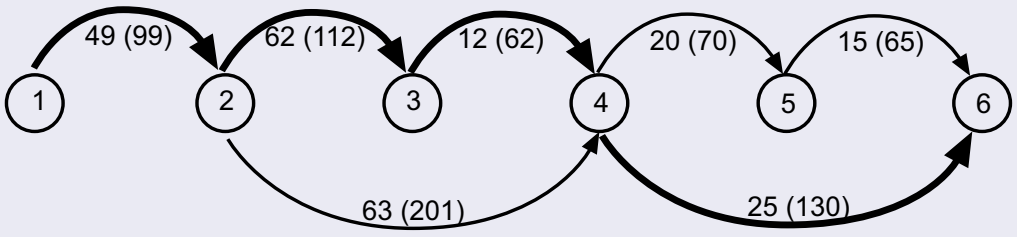

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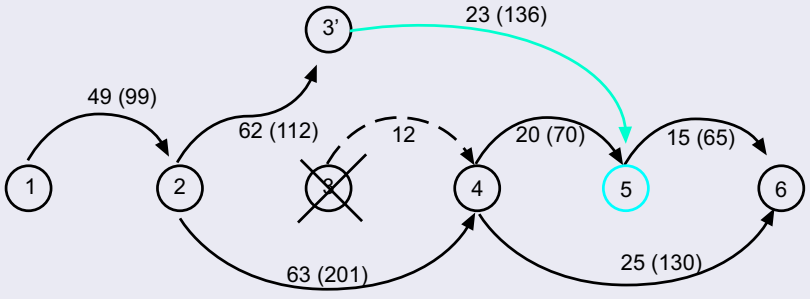

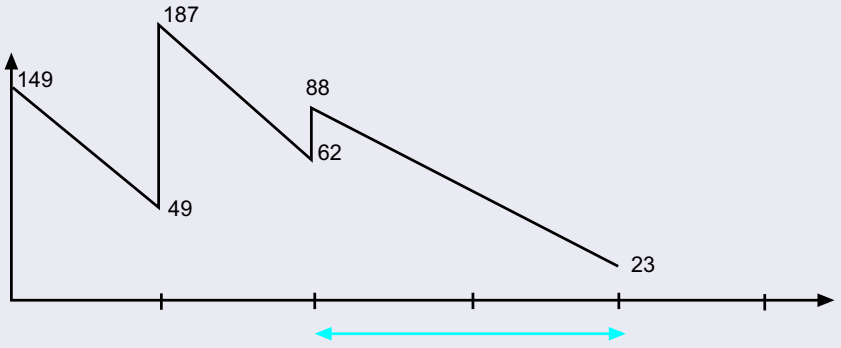

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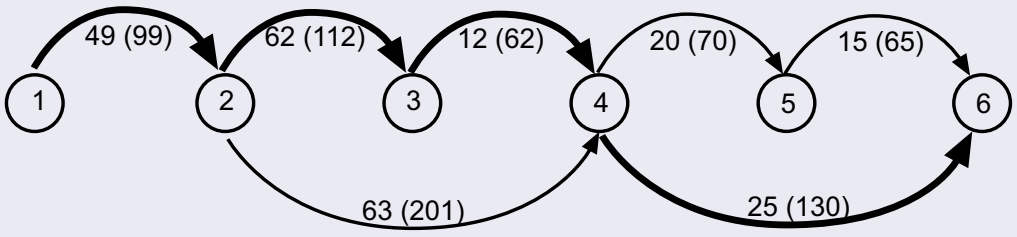

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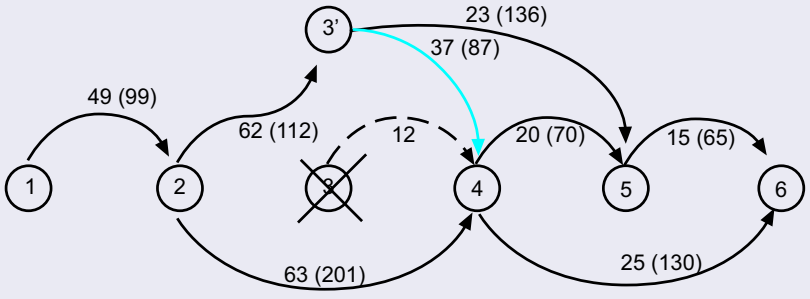

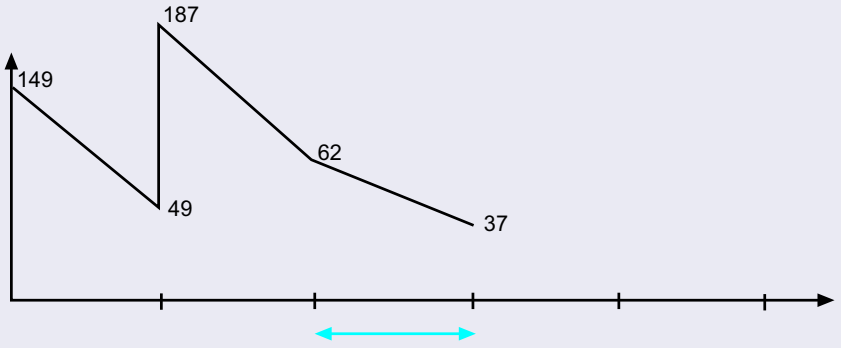

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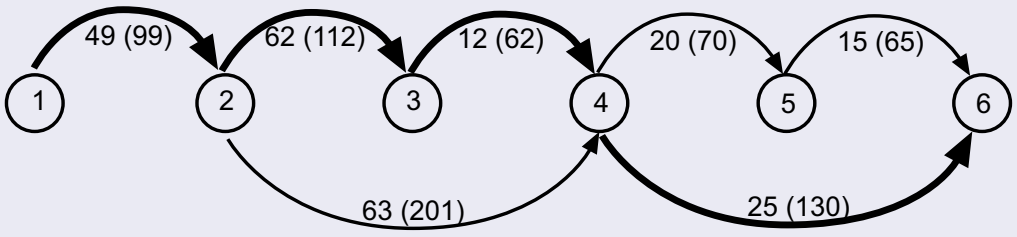

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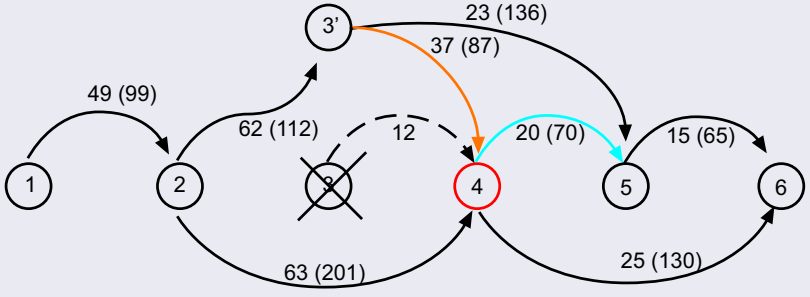

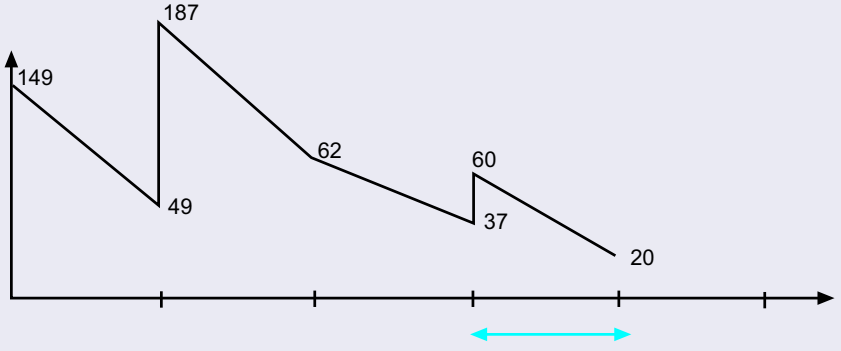

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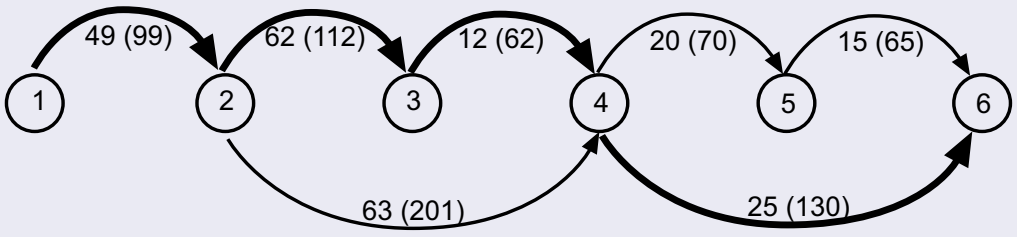

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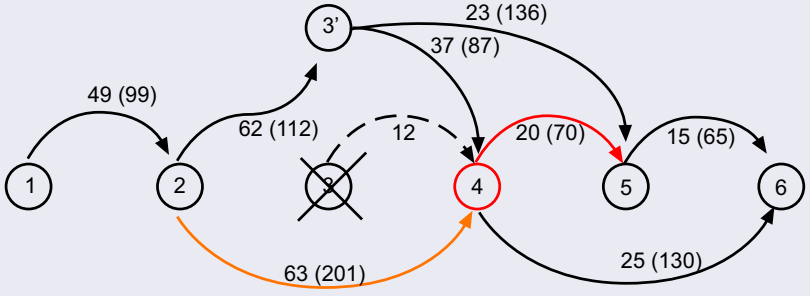

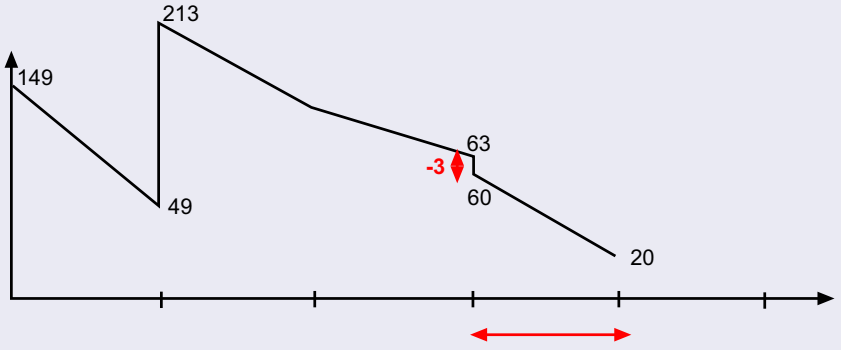

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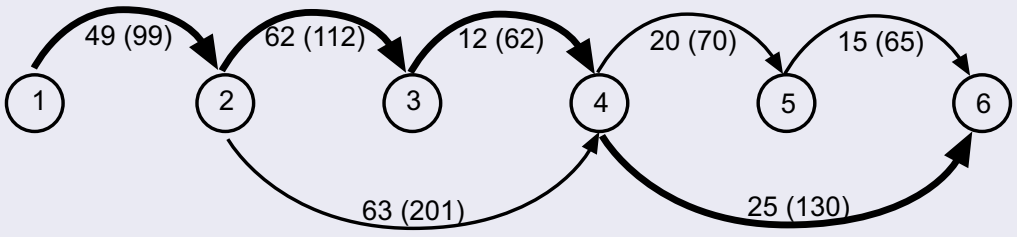

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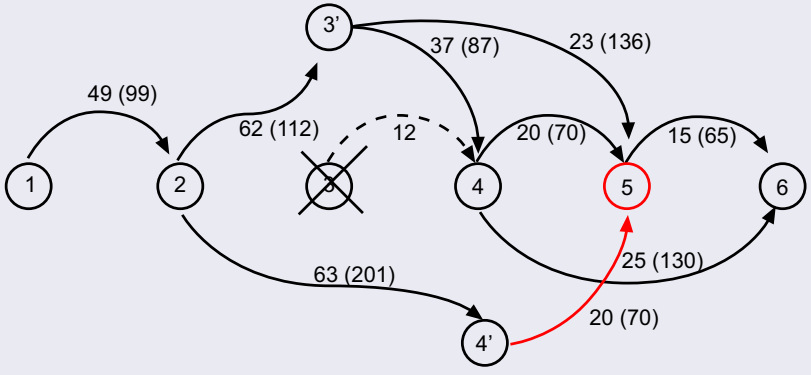

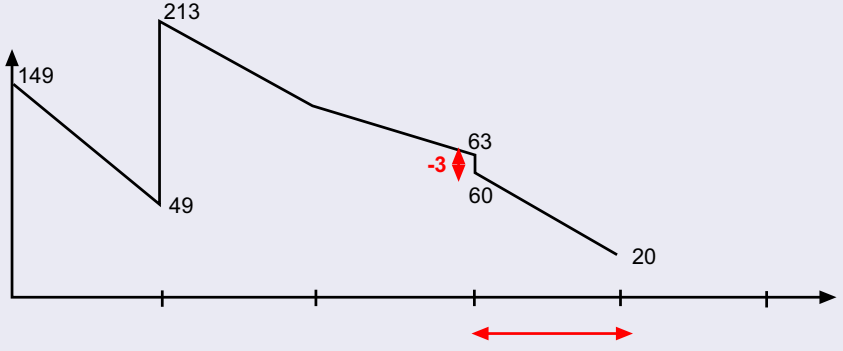

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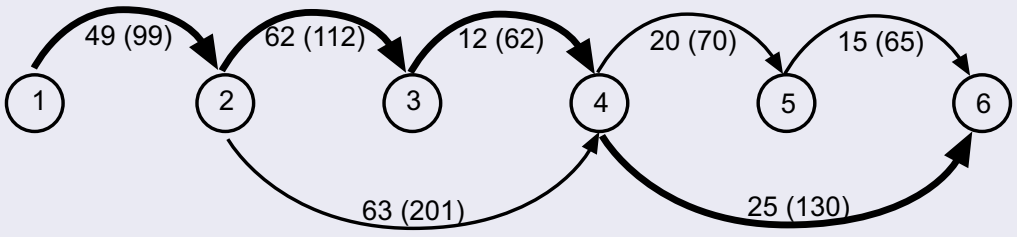

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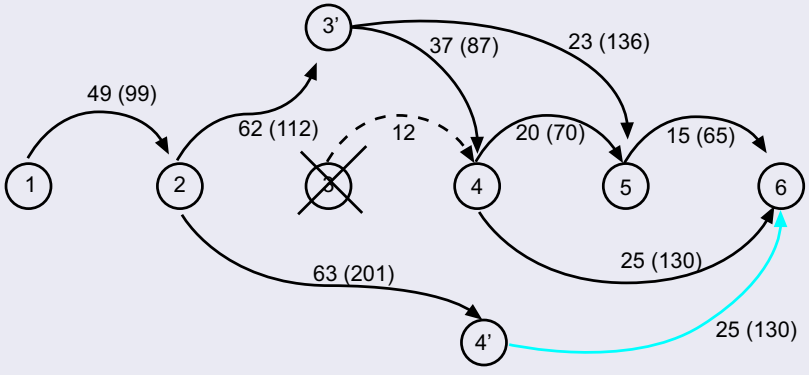

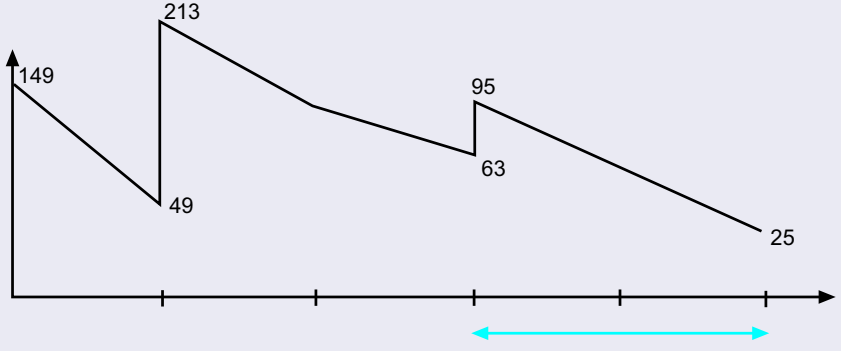

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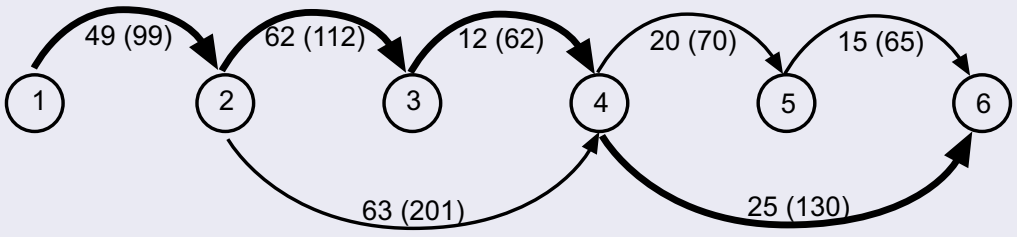

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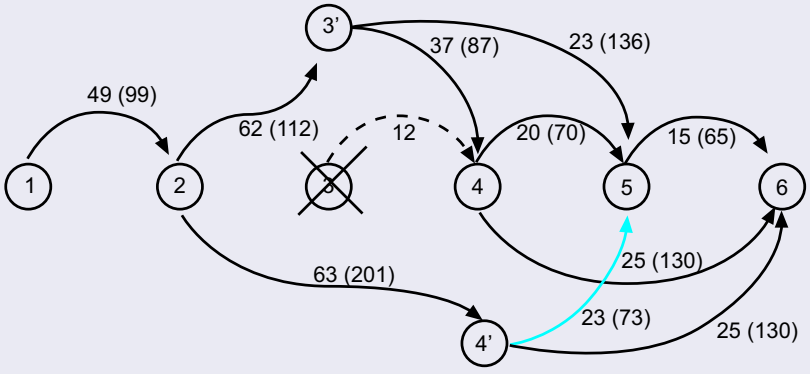

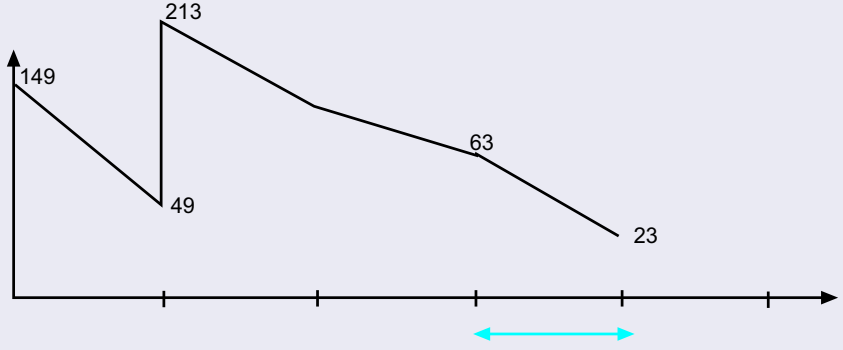

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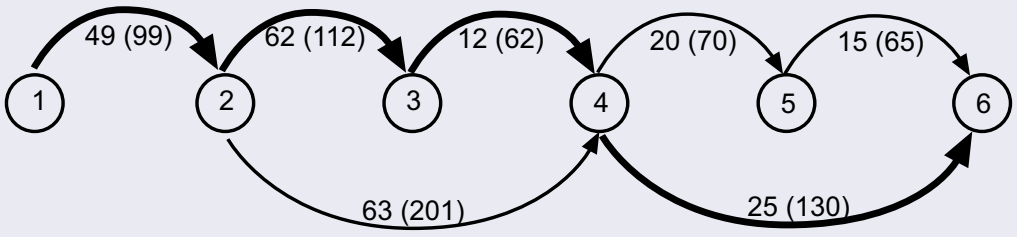

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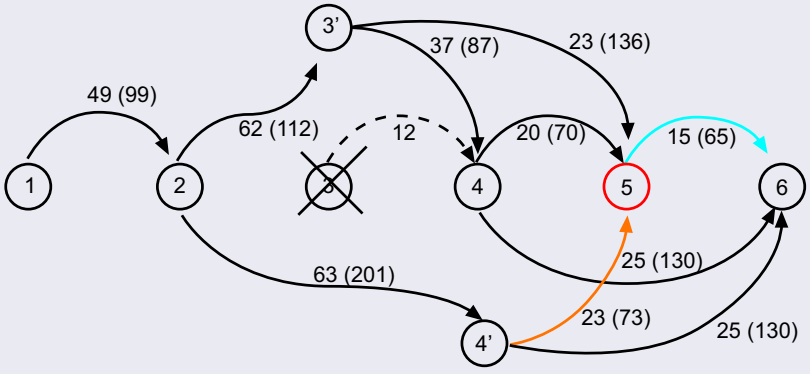

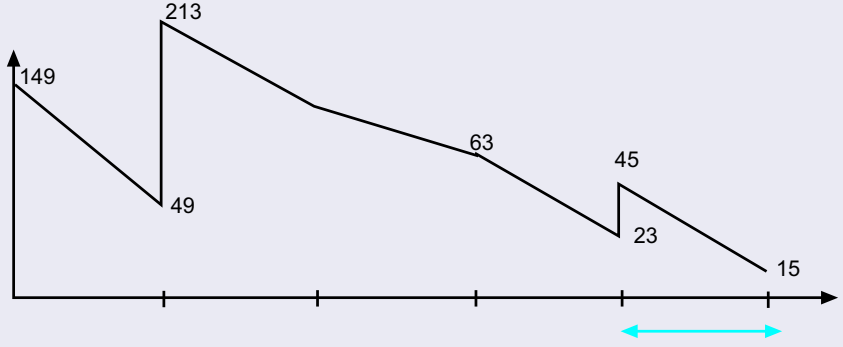

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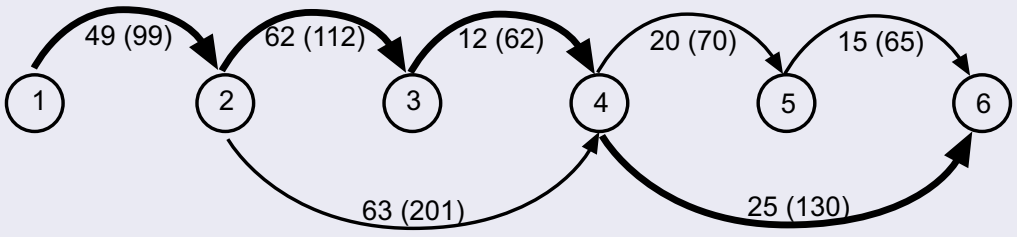

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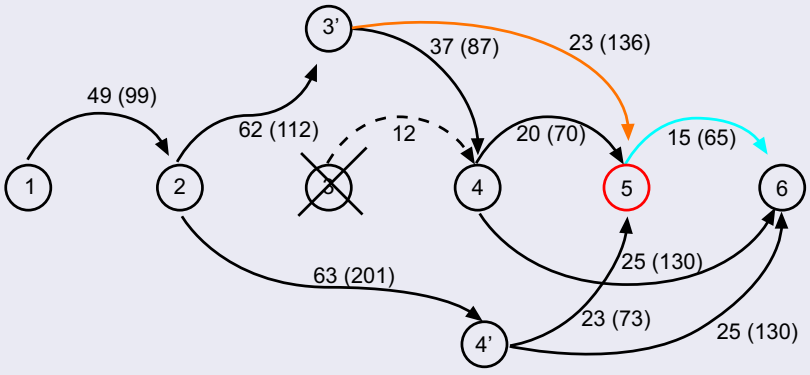

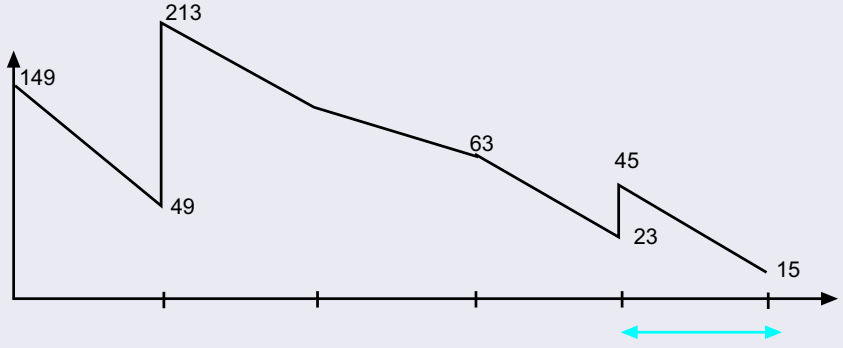

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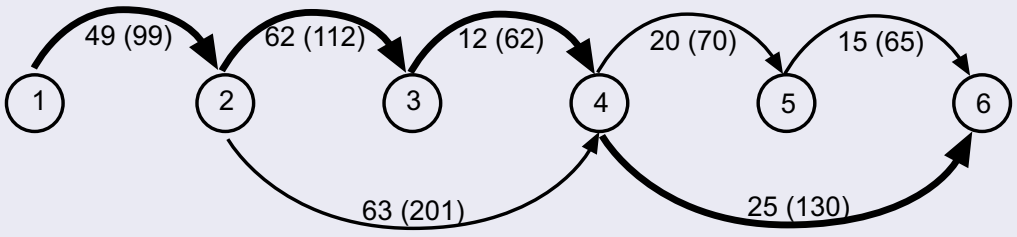

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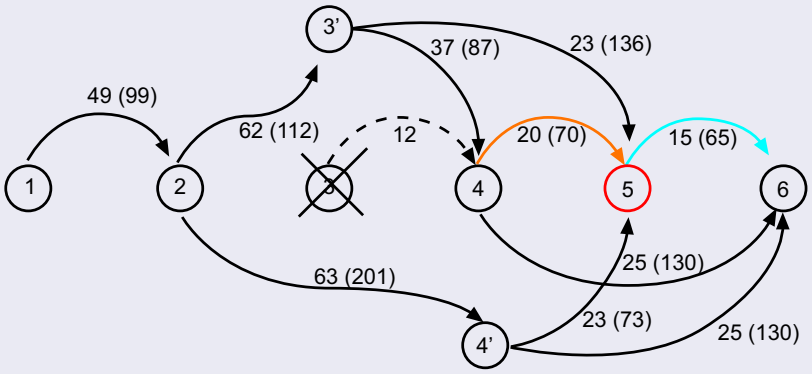

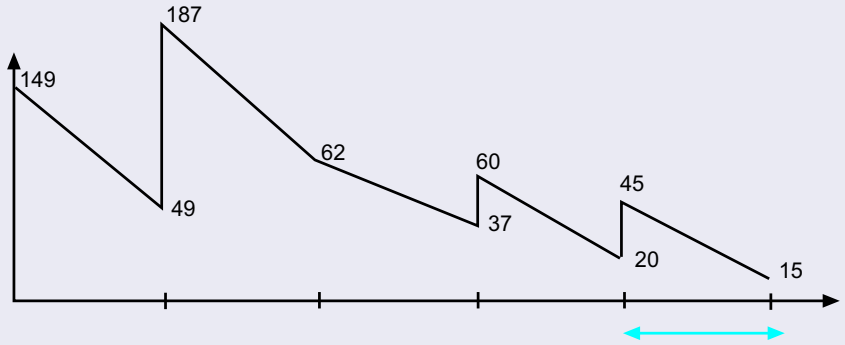

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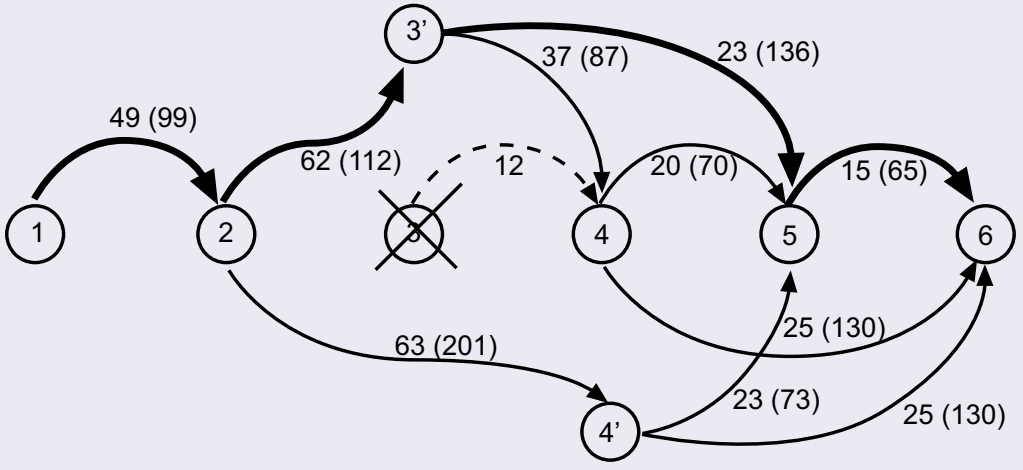
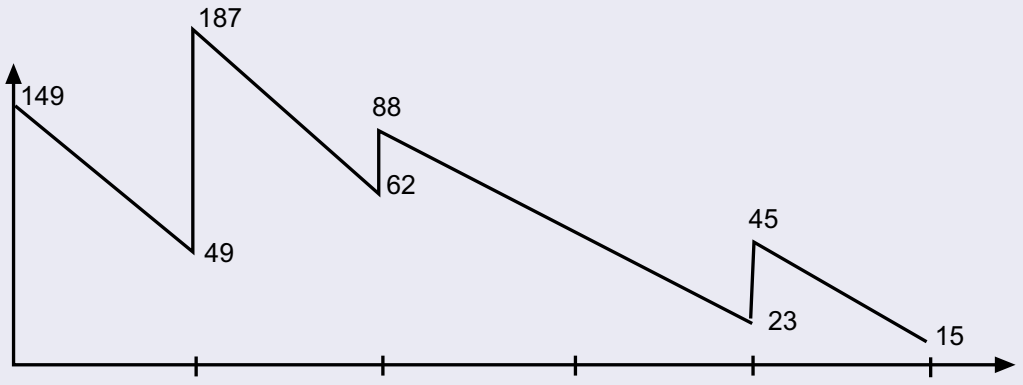


Augmented connection matrix. Expected buffer stock levels and expected cycle costs (in parentheses) are shown for each arc. The shortest path is highlighted. Node 3 (and obviously arc $(3, 4)$) has been removed from the network since the augmenting procedure removed all its inbound arcs

- The shortest path in this augmented network has a cost of 412 and represents the optimal solution cost of our original problem
- The order periods and the order quantities are therefore respectively $[1, 2, 3, 5]$ and $[149, 138, 26, 22]$

Augment



The optimal replenishment plan corresponding to the shortest path computed

- The shortest path in this augmented network has a cost of 412 and represents the optimal solution cost of our original problem
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Input data - 1

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- We consider expected period demands, $\tilde{d}_t$, generated as uniformly distributed random numbers in $[0, 100]$
- fixed ordering cost $a \in \{25, 50, 100, 200\}$
- holding cost $h = 1$ per unit per period
- coefficient of variation $\sigma_t / \tilde{d}_t \in \{1/3, 1/6\}$
- The planning horizon length takes the following values $\{50, 75\}$
- service levels $\alpha = 0.95$ ($z_{\alpha=0.95} = 1.645$) and $\alpha = 0.99$ ($z_{\alpha=0.99} = 2.326$)

Experiments - 1

		$\sigma_t/\tilde{d}_t = 1/3$							
		$\alpha = 0.95$				$\alpha = 0.99$			
		CP		DP		CP		DP	
a	N	Nod	Sec	Graph	Sec	Nod	Sec	Graph	Sec
25	50	857	45	$\langle 64; 76\rangle$	0.33	2474	170	$\langle 67; 82\rangle$	0.30
	75	41386	5412	$\langle 102; 125\rangle$	0.38	180000*	20000*	$\langle 106; 133\rangle$	0.37
50	50	441	17	$\langle 66; 84\rangle$	0.34	1242	170	$\langle 69; 88\rangle$	0.33
	75	23805	2400	$\langle 104; 133\rangle$	0.37	180000*	20000*	$\langle 108; 139\rangle$	0.37
100	50			$\langle 51; 87\rangle$	0.21	104	5	$\langle 73; 109\rangle$	0.34
	75			$\langle 76; 134\rangle$	0.24	329	30	$\langle 113; 167\rangle$	0.21
200	50			$\langle 51; 139\rangle$	0.23			$\langle 51; 131\rangle$	0.24
	75			$\langle 76; 212\rangle$	0.26			$\langle 76; 200\rangle$	0.28

		$\sigma_t/\tilde{d}_t = 1/6$							
		$\alpha = 0.95$				$\alpha = 0.99$			
		CP		DP		CP		DP	
a	N	Nod	Sec	Graph	Sec	Nod	Sec	Graph	Sec
25	50	22	1	$\langle 58; 65\rangle$	0.34	325	17	$\langle 61; 70\rangle$	0.33
	75	245	35	$\langle 90; 103\rangle$	0.2	10118	970	$\langle 98; 116\rangle$	0.20
50	50			$\langle 51; 69\rangle$	0.20	70	3	$\langle 63; 80\rangle$	0.42
	75			$\langle 76; 106\rangle$	0.14	155	14	$\langle 100; 126\rangle$	0.37
100	50			$\langle 51; 103\rangle$	0.13			$\langle 51; 94\rangle$	0.12
	75			$\langle 76; 161\rangle$	0.26			$\langle 76; 145\rangle$	0.25
200	50			$\langle 51; 158\rangle$	0.22			$\langle 51; 184\rangle$	0.15
	75			$\langle 76; 242\rangle$	0.27			$\langle 76; 226\rangle$	0.26

Table: Test set P5. A figure marked with * means that the instance could not be solved in the given limit of 20000 seconds (5.55 hours)

Input data - 2

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- The expected period demands $\tilde{d}_t$ are generated as uniformly distributed random numbers in $[0, 10000]$
- fixed ordering cost $a \in \{2500, 5000, 10000, 20000\}$
- holding cost $h = 1$ per unit per period
- coefficient of variation $\sigma_t / \tilde{d}_t \in \{1/3, 1/6\}$
- The planning horizon length now ranges up to 250 periods
- service levels $\alpha = 0.95$ ($z_{\alpha=0.95} = 1.645$) and $\alpha = 0.99$ ($z_{\alpha=0.99} = 2.326$)

Experiments - 2

		$\sigma_t/\tilde{d}_t = 1/3$				$\sigma_t/\tilde{d}_t = 1/6$			
		$\alpha = 0.95$		$\alpha = 0.99$		$\alpha = 0.95$		$\alpha = 0.99$	
a	N	Graph	Sec	Graph	Sec	Graph	Sec	Graph	Sec
2500	75	$\langle 102; 125\rangle$	0.44	$\langle 106; 133\rangle$	0.38	$\langle 90; 103\rangle$	0.25	$\langle 98; 116\rangle$	0.26
	110	$\langle 153; 194\rangle$	0.36	$\langle 160; 208\rangle$	0.36	$\langle 134; 157\rangle$	0.31	$\langle 144; 175\rangle$	0.28
	145	$\langle 197; 246\rangle$	0.41	$\langle 208; 266\rangle$	0.41	$\langle 174; 202\rangle$	0.41	$\langle 185; 222\rangle$	0.41
	180	$\langle 242; 301\rangle$	0.48	$\langle 255; 327\rangle$	0.49	$\langle 211; 245\rangle$	0.46	$\langle 225; 268\rangle$	0.47
	215	$\langle 284; 350\rangle$	0.96	$\langle 297; 376\rangle$	0.93	$\langle 248; 285\rangle$	0.93	$\langle 263; 310\rangle$	0.88
	250	$\langle 329; 407\rangle$	0.79	$\langle 346; 439\rangle$	0.79	$\langle 287; 330\rangle$	0.75	$\langle 305; 361\rangle$	0.77
5000	75	$\langle 104; 133\rangle$	0.20	$\langle 108; 139\rangle$	0.48	$\langle 76; 107\rangle$	0.13	$\langle 100; 126\rangle$	0.22
	110	$\langle 155; 202\rangle$	0.57	$\langle 162; 214\rangle$	0.29	$\langle 111; 152\rangle$	0.18	$\langle 146; 185\rangle$	0.28
	145	$\langle 199; 255\rangle$	0.36	$\langle 210; 272\rangle$	0.40	$\langle 146; 198\rangle$	0.21	$\langle 187; 234\rangle$	0.62
	180	$\langle 245; 317\rangle$	0.46	$\langle 258; 337\rangle$	0.55	$\langle 181; 250\rangle$	0.30	$\langle 230; 295\rangle$	0.49
	215	$\langle 287; 366\rangle$	0.85	$\langle 300; 386\rangle$	0.94	$\langle 216; 296\rangle$	0.75	$\langle 268; 338\rangle$	0.49
	250	$\langle 332; 426\rangle$	0.76	$\langle 349; 450\rangle$	0.74	$\langle 251; 347\rangle$	0.61	$\langle 312; 399\rangle$	0.95
10000	75	$\langle 76; 134\rangle$	0.13	$\langle 116; 174\rangle$	0.34	$\langle 76; 162\rangle$	0.15	$\langle 76; 147\rangle$	0.04
	110	$\langle 170; 270\rangle$	0.29	$\langle 171; 256\rangle$	0.19	$\langle 111; 230\rangle$	0.22	$\langle 111; 211\rangle$	0.10
	145	$\langle 216; 344\rangle$	0.62	$\langle 224; 332\rangle$	0.35	$\langle 146; 300\rangle$	0.26	$\langle 146; 280\rangle$	0.19
	180	$\langle 271; 439\rangle$	0.51	$\langle 279; 422\rangle$	0.69	$\langle 181; 377\rangle$	0.34	$\langle 181; 354\rangle$	0.25
	215	$\langle 317; 517\rangle$	0.85	$\langle 324; 493\rangle$	0.61	$\langle 216; 448\rangle$	0.70	$\langle 216; 423\rangle$	0.58
	250	$\langle 365; 593\rangle$	1.02	$\langle 375; 569\rangle$	0.99	$\langle 251; 512\rangle$	0.62	$\langle 251; 485\rangle$	0.67
20000	75	$\langle 76; 212\rangle$	0.14	$\langle 76; 201\rangle$	0.15	$\langle 76; 242\rangle$	0.15	$\langle 76; 228\rangle$	0.17
	110	$\langle 111; 306\rangle$	0.21	$\langle 111; 292\rangle$	0.13	$\langle 111; 352\rangle$	0.17	$\langle 111; 332\rangle$	0.16
	145	$\langle 146; 408\rangle$	0.27	$\langle 146; 388\rangle$	0.24	$\langle 146; 460\rangle$	0.39	$\langle 146; 437\rangle$	0.26
	180	$\langle 181; 514\rangle$	0.35	$\langle 181; 485\rangle$	0.49	$\langle 181; 575\rangle$	0.38	$\langle 181; 546\rangle$	0.54
	215	$\langle 216; 617\rangle$	0.67	$\langle 216; 585\rangle$	0.44	$\langle 216; 685\rangle$	0.50	$\langle 216; 652\rangle$	0.47
	250	$\langle 251; 713\rangle$	0.68	$\langle 251; 675\rangle$	0.60	$\langle 251; 787\rangle$	0.80	$\langle 251; 750\rangle$	0.79

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- The concept of *state space augmentation* [2] is also known in the Operations Research literature. A dual approach to state space augmentation is *decremental SSR* (Righini & Salani [10])
- Nevertheless, the idea of filtering a relaxed state space graph is, to the best of our knowledge, a novel contribution.

The End

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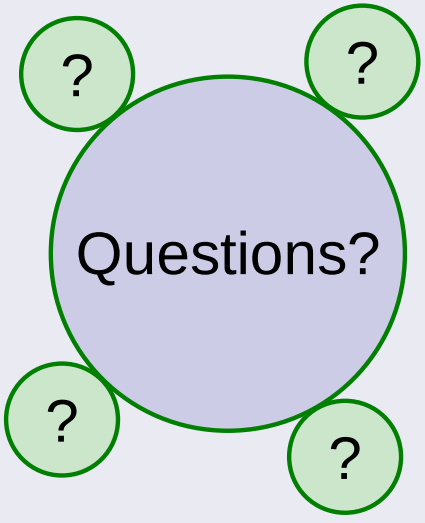

Acknowledgments: this work was supported by Science Foundation Ireland under Grant No. 03/CE3/I405 as part of the Centre for Telecommunications Value-Chain Research (CTVR) and Grant No. 05/IN/I886. S. Armagan Tarim and Brahim Hnich are supported by the Scientific and Technological Research Council of Turkey (TUBITAK) under Grant No. SOBAG-108K027

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