An Efficient Computational Method for Non-Stationary (R,S) Inventory Policy with Service Level Constraints

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System Under Study

- Single-item, single stage lot sizing problem
- Supplier with ample capacity
- No lead time (Without loss of generality)
- Periodic review (system state and actions)
  - planning horizon: N periods
- Stochastic demand of the customer
  - non-stationary demand: $D_t \geq 0$, $t=1,2,...,N$
- Fixed cost of ordering (A)
- Variable cost of holding inventory (h)
- Service level constraints (P1: probability of stock outs)
Sequence of Events

At the start of the planning horizon

Knowing the demand forecasts, the planner determines timing of the orders, and the fixed ordering costs are incurred.

In each period

If it is a replenishment period, the planner decides on the order quantity to achieve certain service level after evaluating the current inventory position. The order due to arrive is received, and demand occurs.

Holding cost is incurred on the period ending inventory.
Non-Stationary (R,S) Policy

(R,S) Policy:

- (R,S) pairs change within the planning horizon
- m orders → (R_i,S_i) for i=1,2,...,m
Model

**Decision Variables:** Replenishment periods, $\delta_t \in \{0, 1\}$

Target inventory levels $S_t \geq 0$

**Objective:** Minimize the expected ordering and inventory holding costs within the planning horizon while satisfying a service level constraint in each period

\[
\min \mathbb{E} \left[ \sum_{t=1}^{N} \left( A\delta_t + h \max\{0, I_t\} \right) \right]
\]

**Inventory Balance Equations**

\[
\begin{align*}
I_t &= \max\{S_t, I_{t-1}\} - D_t & \text{if } & \delta_t = 1 \\
I_t &= I_{t-1} - D_t & \text{if } & \delta_t = 0
\end{align*}
\]

**Service Level Constraints**

\[
\Pr\{I_t \geq 0\} \geq \alpha
\]

Objective Function

Inventory Balance Equations

Service Level Constraints
Literature & Model Assumptions

- Bookbinder and Tan (Management Science, 1988)
  - Fix the replenishment schedule with a heuristic: $\delta_t \in \{0,1\}$
  - Work with expectations rather than random variables
    \[
    \min \sum_{t=1}^{N} \left( A\delta_t + h\tilde{I}_t \right)
    \]
    \[
    \tilde{I}_t = \max\{\tilde{S}_t, \tilde{I}_{t-1} \} - \tilde{d}_t \quad \text{if} \quad \delta_t = 1
    \]
    \[
    \tilde{I}_t = \tilde{I}_{t-1} - \tilde{d}_t \quad \text{if} \quad \delta_t = 0
    \]
    \[
    \tilde{I}_i \geq G^{-1}_i(\alpha) - \tilde{d}_i, \quad i = 1, 2, \ldots, m
    \]

- Tarim and Kingsman (IJPE, 2004)
  - Under the same model assumptions of BT(1988), formulate a MIP model that also determines the replenishment schedule
MIP Formulation of Tarim and Kingsman (2004)

\[
\begin{align*}
\min \ E[TC'] &= \sum_{i=1}^{N} \left( A\delta_i + h\tilde{I}_i \right) \\
\text{s.t.} \\
\tilde{I}_i &= \tilde{S}_i - \tilde{d}_i \\
\tilde{S}_i &\geq \tilde{I}_{i-1} \\
\tilde{S}_i - \tilde{I}_{i-1} &\leq M\delta_i \\
\tilde{I}_i &\geq \sum_{j=1}^{t} \left( G^{-1}_{d_{i-j+1}+d_{i-j+2}+\ldots+d_{i}}(\alpha_i) - \sum_{k=t-j+1}^{t} \tilde{d}_k \right) P_{tj} \\
\sum_{j=1}^{t} P_{tj} &= 1 \\
\tilde{I}_i &\geq 0 \\
\tilde{P}_{tj} &\geq \delta_{t-j+1} - \sum_{k=t-j+2}^{t} \delta_k \\
\delta_i, P_{tj} &\in \{0, 1\} \\
\end{align*}
\]
Solution of the MIP Model for a Given Replenishment Schedule

Let \( \mathcal{I} \) and \( \mathcal{I}^c \): \( \mathcal{I} \cup \mathcal{I}^c = \{1, \ldots, N\} \), \( \delta_i = 1 \) for all \( i \in \mathcal{I} \) and \( \delta_i = 0 \) for all \( i \in \mathcal{I}^c \).

**Lemma 1** Given any \( \mathcal{I} \) and \( \mathcal{I}^c \), the optimal solution for the MIP model has

\[
\tilde{S}_t = \begin{cases} 
\max \left\{ \tilde{S}_{t-1} - \tilde{d}_{t-1}, G_{d_{t} + d_{t+1} + \ldots + d_{t-1}}^{-1} (\alpha), \sum_{k=t}^{t-1} \tilde{d}_k \right\} & \text{for } t \in \mathcal{I} \\
\tilde{S}_{t-1} - \tilde{d}_{t-1} & \text{for } t \in \mathcal{I}^c,
\end{cases}
\]
Equivalent Optimization Problem

The result of Lemma 1 can be used to formulate an equivalent optimization problem for the MIP

$$\min_{\delta_1, \ldots, \delta_N} z = \sum_{t=1}^{N} \left( a\delta_t + h(\tilde{S}_t - \tilde{d}_t) \right)$$
Relaxed MIP Model

\[
\begin{align*}
\min \quad & E[TC] = \sum_{t=1}^{N} \left( A\delta_t + h\tilde{I}_t \right) \\
\text{s.t.} \quad & \tilde{I}_t = \tilde{S}_t - \tilde{d}_t \\
& \tilde{S}_t \geq \tilde{I}_{t-1} \\
& \tilde{S}_t - \tilde{I}_{t-1} \leq M\delta_t \\
& \tilde{I}_t \geq \sum_{j=1}^{t} \left( G_{d_{t-j+1} + d_{t-j+2} + \ldots + d_t}^{-1}(\alpha_t) - \sum_{k=t-j+1}^{t} \tilde{d}_k \right) P_{tj} \\
& \sum_{j=1}^{t} P_{tj} = 1 \\
& \tilde{I}_t \geq 0 \\
& P_{tj} \geq \delta_{t-j+1} - \sum_{k=t-j+2}^{t} \delta_k \\
& \delta_t, P_{tj} \in \{0, 1\}
\end{align*}
\]
Solution of the Relaxed MIP Model for a Given Replenishment Schedule

\[ \mathcal{I} \text{ and } \mathcal{I}^C: \mathcal{I} \cup \mathcal{I}^C = \{1, \ldots, N\}, \delta_i = 1 \text{ for all } i \in \mathcal{I} \text{ and } \delta_i = 0 \text{ for all } i \in \mathcal{I}^C \]

**Lemma 2** Given any \( \mathcal{I} \) and \( \mathcal{I}^C \), the optimal solution for the relaxed MIP model has

\[
\tilde{S}_t^r = \begin{cases}
G_{d_t + d_{t+1} + \ldots + d_{t-1}}(\alpha) & \text{for } t \in \mathcal{I} \\
\tilde{S}_{t-1}^r - \tilde{d}_{t-1} & \text{for } t \in \mathcal{I}^C,
\end{cases}
\]

Equivalent Optimization Problem

\[
\min_{\delta_1, \ldots, \delta_N} z_r = \sum_{t=1}^{N} \left( a\delta_t + h(\tilde{S}_t^r - \tilde{d}_t) \right)
\]
Overview of Our Results

\[ \min_{\delta_1, \ldots, \delta_N} z = \sum_{t=1}^{N} \left( a\delta_t + h(\tilde{S}_t - \tilde{d}_t) \right) \]

true \quad \text{false} \quad z_r^* \leq z^*

Lemma 1

Relaxed MIP

Feasibility Check

\{ \tilde{S}_t | \delta_1^*, \ldots, \delta_N^* \} = \tilde{S}_t^{r*}

MIP

Theorem

Equivalent Shortest Path Formulation

\[ \delta^* = (\delta_1^*, \ldots, \delta_N^*), S^* = (\tilde{S}_1^{r*}, \ldots, \tilde{S}_N^{r*}), z_r^* \]

\[ O(|A|), |A| = \frac{N(N+1)}{2} \]

Lemma 2

\[ \min_{\delta_1, \ldots, \delta_N} z_r = \sum_{t=1}^{N} \left( a\delta_t + h(\tilde{S}_t^r - \tilde{d}_t) \right) \]
Computational Procedure

- Solve the shortest path problem $\rightarrow$ optimal solution for the Relaxed MIP

$$\delta^* = (\delta_1^*, \ldots, \delta_N^*), \ S^* = (\tilde{S}_1^*, \ldots, \tilde{S}_N^*), \ z^*$$

- Check the feasibility of the solution
  - Feasible $\rightarrow$ Terminate
  - Not feasible
    - Lower Bound ($z_r^*$)
    - Use $\delta^*$ to calculate $\tilde{S}_t \rightarrow$ Upper bound ($z$)
    - Initiate a Branch & Bound search and branch on one $\delta_t$ that is not feasible
Numerical Study

We would like to address

- the percentage of the non-stationary instances solved to optimality using solely the relaxed-MIP approach, without resorting to any search effort
- the effectiveness of the bounds provided by the relaxed-MIP model if the observed solution is infeasible for the original problem
- the overall solution time performance of the proposed method
- the scalability of the proposed method

Test bed

- N=30, 40, 50, 60
- Demand Patterns
  - $\mu_t$: stationary ($P_1$), seasonal ($P_2$), decreasing ($P_3$), increasing ($P_4$), product life-cycle ($P_5$)
  - $D_t \sim \text{Normal}(r_t \mu_t, (0.25r_t \mu_t)^2)$ where $r_t$ is sampled from $U[0.4, 1.6]$
  - $h=1, \alpha=0.95, A$ is sampled from $U[75, 2000]$
Numerical Results

Generated random instances and solved using the relaxed MIP approach (equivalent shortest path formulation) for a given $N$ and a demand pattern ($P_i$)

<table>
<thead>
<tr>
<th></th>
<th>$P_1$</th>
<th>$P_2$</th>
<th>$P_3$</th>
<th>$P_4$</th>
<th>$P_5$</th>
</tr>
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<tbody>
<tr>
<td>$N = 30$</td>
<td>-</td>
<td>124,101</td>
<td>238,794</td>
<td>304,239</td>
<td>49,279</td>
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<tr>
<td>$N = 40$</td>
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<td>104,686</td>
<td>134,702</td>
<td>172,369</td>
<td>54,918</td>
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<tr>
<td>$N = 50$</td>
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<td>62,780</td>
<td>223,889</td>
<td>151,011</td>
<td>22,251</td>
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<tr>
<td>$N = 60$</td>
<td>-</td>
<td>53,086</td>
<td>246,614</td>
<td>128,831</td>
<td>31,400</td>
</tr>
</tbody>
</table>

2,102,950 instances
160 infeasibility
99.99%

Infeasible problems

- Java Implementation, CPLEX 11.2, 2.0 GHz CPU with 32-bit machine
- Cut-off time of 1 hour
## Numerical Results (cont.)

<table>
<thead>
<tr>
<th>Demand</th>
<th>N = 30</th>
<th>Our Method</th>
<th>MIP</th>
<th>N = 40</th>
<th>Our Method</th>
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</thead>
<tbody>
<tr>
<td></td>
<td>Nodes</td>
<td>%Δ</td>
<td>%Δ&lt;sub&gt;LB&lt;/sub&gt;</td>
<td>%Δ&lt;sub&gt;UB&lt;/sub&gt;</td>
<td>sccs</td>
</tr>
<tr>
<td>1</td>
<td>19200</td>
<td>12.4</td>
<td>0.00</td>
<td>0.00</td>
<td>0.1</td>
</tr>
<tr>
<td>2</td>
<td>21900</td>
<td>12.2</td>
<td>0.12</td>
<td>0.16</td>
<td>0.7</td>
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<tr>
<td>3</td>
<td>48000</td>
<td>24.5</td>
<td>0.00</td>
<td>0.00</td>
<td>0.0</td>
</tr>
<tr>
<td>4</td>
<td>9000</td>
<td>6.7</td>
<td>0.00</td>
<td>0.00</td>
<td>0.0</td>
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<tr>
<td>5</td>
<td>36100</td>
<td>21.8</td>
<td>0.00</td>
<td>0.00</td>
<td>0.0</td>
</tr>
<tr>
<td>6</td>
<td>11500</td>
<td>7.6</td>
<td>0.08</td>
<td>0.00</td>
<td>1.4</td>
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<tr>
<td>7</td>
<td>23400</td>
<td>14.4</td>
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<tr>
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<td>12100</td>
<td>8.6</td>
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<td>0.00</td>
<td>0.8</td>
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<tr>
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<tr>
<td>10</td>
<td>26800</td>
<td>16.3</td>
<td>0.24</td>
<td>0.04</td>
<td>3.0</td>
</tr>
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</table>

**Table:**

- **P2**
  - 22700: Nodes = 59, %Δ = 0.04, %Δ<sub>LB</sub> = 0.04, %Δ<sub>UB</sub> = 0.7, sccs = 51.
  - 66500: Nodes = 91, %Δ = 0.08, %Δ<sub>LB</sub> = 0.08, %Δ<sub>UB</sub> = 0.8, sccs = 52.
- **P3**
  - 28300: Nodes = 107, %Δ = 0.12, %Δ<sub>LB</sub> = 0.08, %Δ<sub>UB</sub> = 1.0, sccs = 53.
  - 30500: Nodes = 59, %Δ = 0.08, %Δ<sub>LB</sub> = 0.00, %Δ<sub>UB</sub> = 0.7, sccs = 54.
- **P4**
  - 16800: Nodes = 59, %Δ = 0.00, %Δ<sub>LB</sub> = 0.00, %Δ<sub>UB</sub> = 0.1, sccs = 61.
  - 11600: Nodes = 59, %Δ = 0.04, %Δ<sub>LB</sub> = 0.00, %Δ<sub>UB</sub> = 0.6, sccs = 62.
- **P5**
  - 25800: Nodes = 59, %Δ = 0.03, %Δ<sub>LB</sub> = 0.07, %Δ<sub>UB</sub> = 0.7, sccs = 71.
  - 17500: Nodes = 59, %Δ = 0.04, %Δ<sub>LB</sub> = 0.00, %Δ<sub>UB</sub> = 0.6, sccs = 72.
## Numerical Results (cont.)

<table>
<thead>
<tr>
<th>Demand</th>
<th>MIP</th>
<th>N = 50</th>
<th>Our Method</th>
<th>N = 60</th>
<th>Our Method</th>
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</thead>
<tbody>
<tr>
<td></td>
<td>N</td>
<td>Nodes</td>
<td>%Δ</td>
<td>%ΔLB</td>
<td>%ΔUB</td>
</tr>
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<td></td>
<td>N</td>
<td>%Δ</td>
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<td>N</td>
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</tbody>
</table>

### P2

- 81: 2360900 4.22
- 82: 2308100 3.00
- 83: 2264900 1.89
- 84: 2570400 5.89
- 85: 2249000 3.60
- 86: 2541500 4.55
- 87: 2651300 2.61
- 88: 2329900 4.22
- 89: 2323100 4.18
- 90: 2442900 3.42

### P3

- 91: 3575200 4.51
- 92: 2907800 6.90
- 93: 2747100 4.35
- 94: 2915200 3.36
- 95: 2792000 4.56
- 96: 2829500 6.62
- 97: 3074500 2.89
- 98: 2770200 3.50
- 99: 2845500 2.24
- 100: 3569400 6.02

### P4

- 101: 2232100 3.55
- 102: 2393000 3.60
- 103: 1294100 2.069
- 104: 2344100 4.28
- 105: 2403900 0.48
- 106: 2463600 1.71
- 107: 2540200 3.45
- 108: 2526500 3.50
- 109: 2329100 2.49
- 110: 2588600 5.08

### P5

- 111: 2802300 1.79
- 112: 2403700 0.19
- 113: 2143500 0.96
- 114: 3040700 2.37
- 115: 1692900 2.310
- 116: 2260700 6.69
- 117: 3120500 3.74
- 118: 937000 13.75
- 119: 2899500 3.54
- 120: 1224000 17.30

### Notes

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