On the stochastic inventory problem under order capacity constraints

Roberto Rossi
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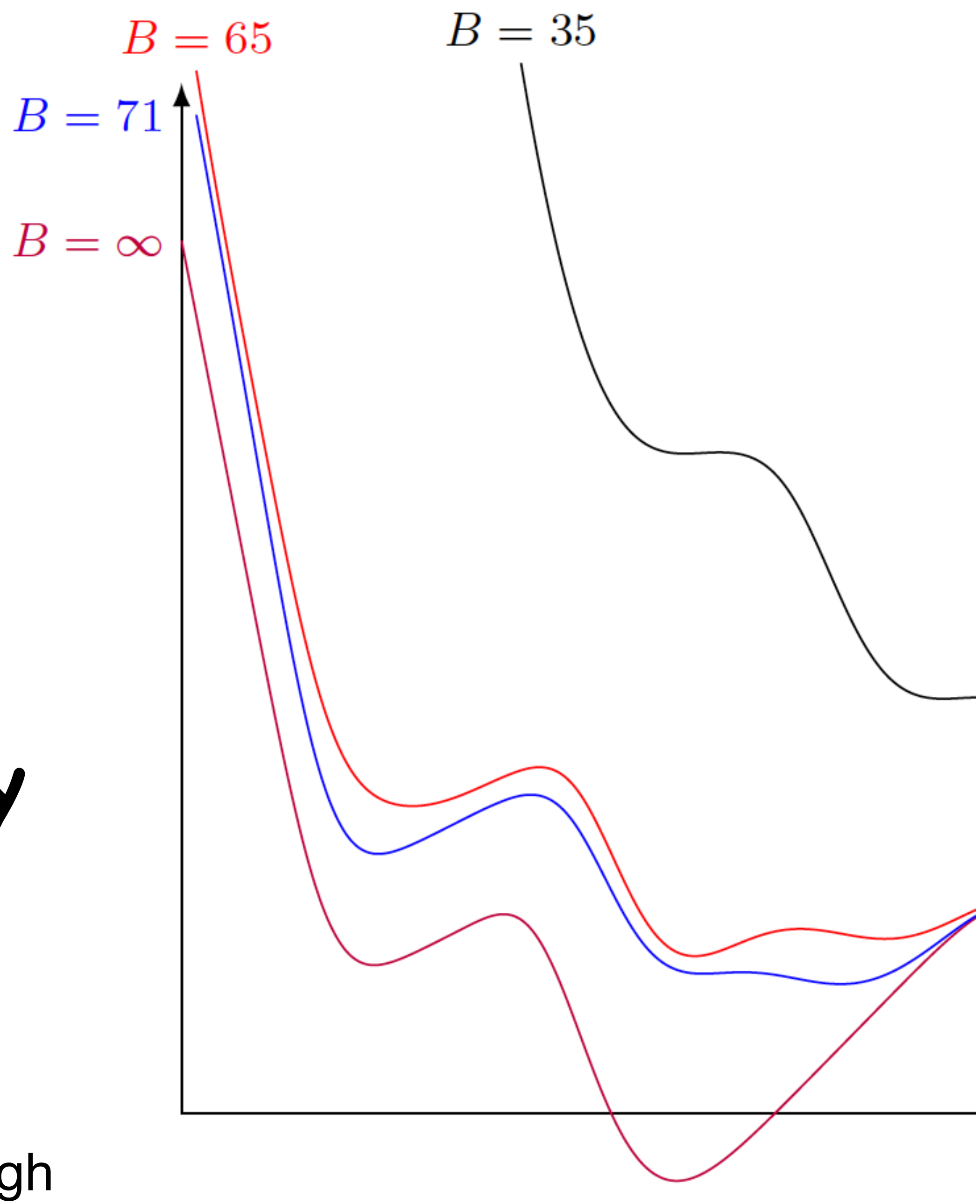

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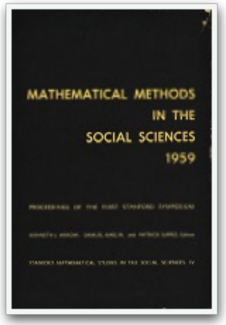

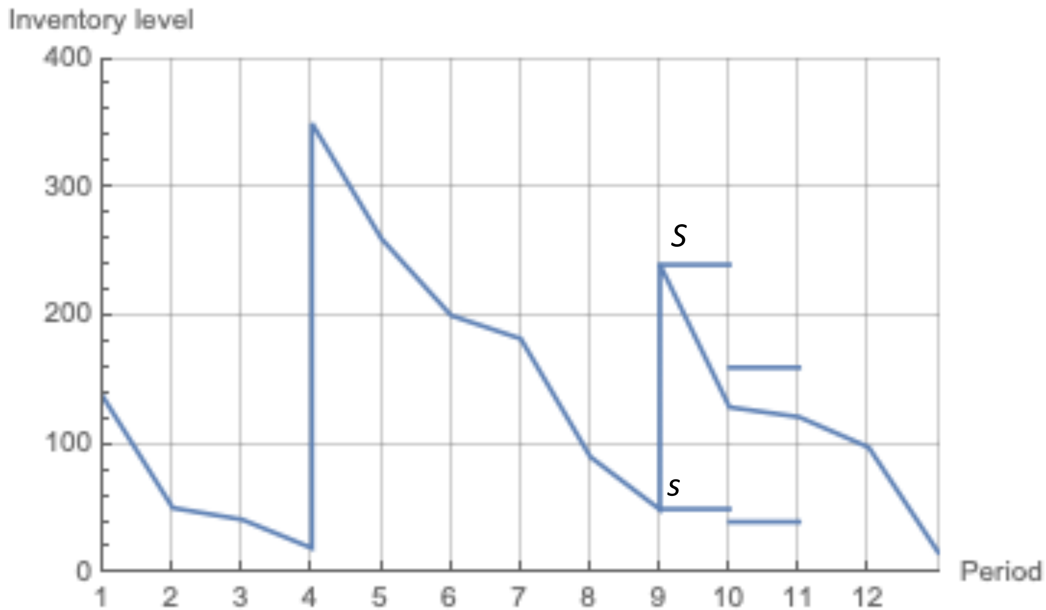

- single-item single-stocking point inventory control problem
- a finite planning horizon of n discrete time periods
- random demand d_t in period t
- ordering cost $c(x)$ for placing an order for x units
- holding cost h for any unit of inventory carried over to next period
- shortage cost p for each unit of unmet demand in any given period

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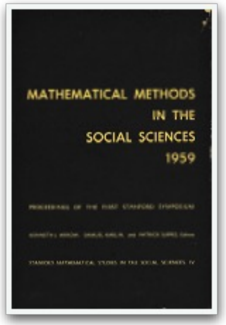

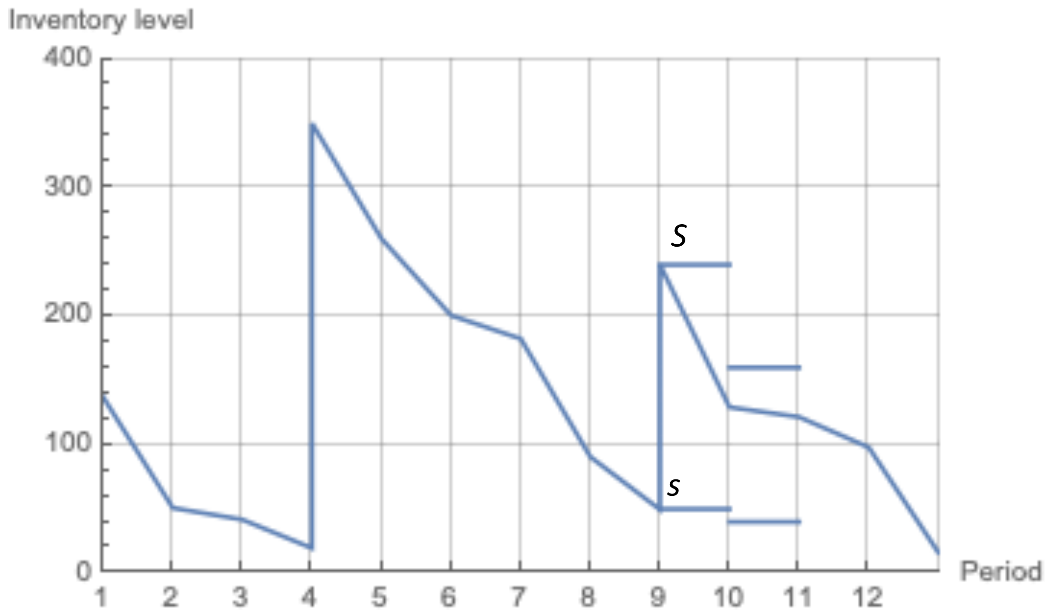

Let x represent the pre-order inventory level, and $\widehat{C}_n(x)$ denote the minimum expected total cost achieved by employing an optimal replenishment policy over the planning horizon $n, \ldots, 1$; then one can write

$$\widehat{C}_n(x) \triangleq \min_{x \leq y} \left\{ c(y-x) + L_n(y) + \int_0^{\infty} \widehat{C}_{n-1}(y-\xi) f_n(\xi) \mathrm{d}\xi \right\},$$

where $\widehat{C}_0 \triangleq 0$ and $L_n(y) \triangleq \int_0^y h(y-\xi) f_n(\xi) \mathrm{d}\xi + \int_y^{\infty} p(\xi - y) f_n(\xi) \mathrm{d}\xi$.

Following Scarf [1960], it is assumed that the ordering cost takes the form

$$c(x) \triangleq \begin{cases} 0 & x = 0, \\ K + vx & x > 0. \end{cases}$$

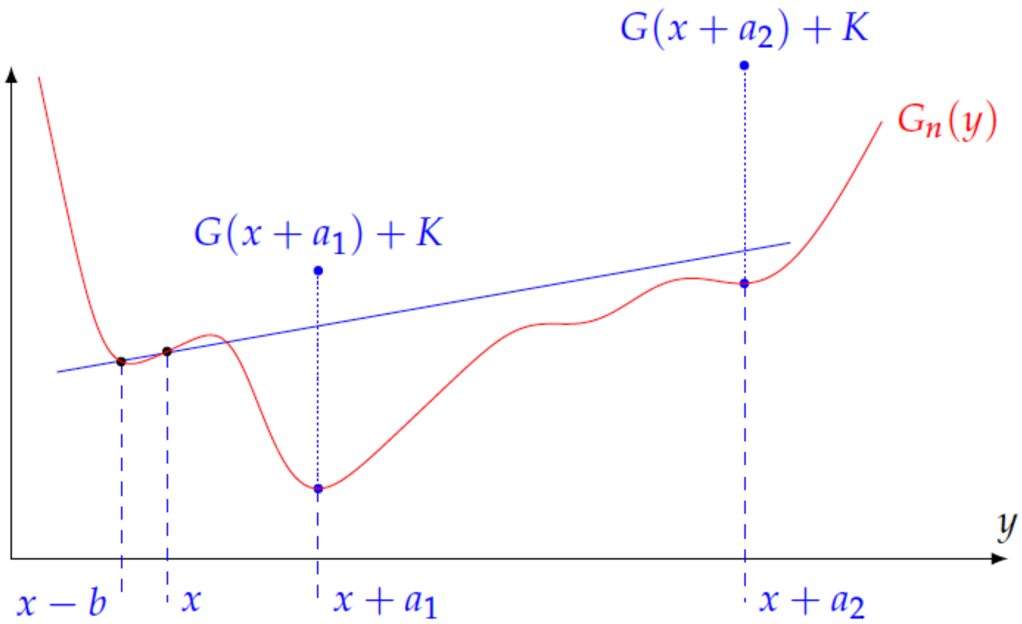

Fig. 98 K-convexity of G_n: let $a, b > 0$, pick two points $(x-b, G(x-b))$ and $(x, G(x))$, draw a straight line passing through them; then for any $x+a$, point $(x+a, G(x+a)+K)$ lies above the straight line.

More specifically, Scarf [1960] introduced the concept of K-convexity (Definition 1).

Definition 1 (K-convexity)**.** *Let $K \geq 0$, $g(x)$ is K-convex if for all x, $a > 0$, and $b > 0$,*

$$(K + g(x+a) - g(x))/a \geq (g(x) - g(x-b))/b;$$

and proved that $\widehat{G}_n(y)$ is K-convex, where

$$\widehat{G}_n(y) \triangleq vy + L_n(y) + \int_0^\infty \widehat{C}_{n-1}(y-\xi) f_n(\xi) \mathrm{d}\xi.$$

This observation implies that the (s, S) policy is optimal, and the policy parameters s and S satisfy $\widehat{G}_n(s) = \widehat{G}_n(S) + K$.

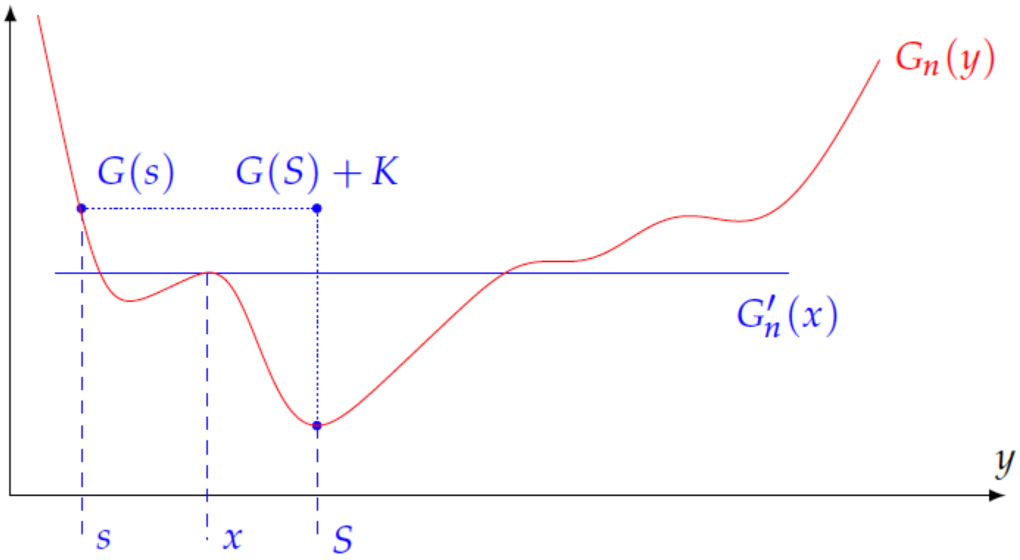

Fig. 99 K-convexity of G_n: $K + G_n(S)$ is greater than the value of G_n at any local maximum $x < S$, thus there exists a unique value s such that $K + G_n(S) = G_n(s)$.

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Production, Manufacturing, Transportation and Logistics

On the stochastic inventory problem under order capacity constraints☆

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A R T I C L E I N F O

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We consider the single-item single-stocking location stochastic inventory system under a fixed ordering cost component. A long-standing problem is that of determining the structure of the optimal control policy when this system is subject to order quantity capacity constraints; to date, only partial characterisations of the optimal policy have been discussed. An open question is whether a policy with a single continuous interval over which ordering is prescribed is optimal for this problem. Under the so-called "continuous order property" conjecture, we show that the optimal policy takes the modified multi-(s, S) form. Moreover, we provide a numerical counterexample in which the continuous order property is violated, and hence show that a modified multi-(s, S) policy is not optimal in general. However, in an extensive computational study, we show that instances violating the continuous order property do not surface, and that the plans generated by a modified multi-(s, S) policy can therefore be considered, from

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The stochastic inventory problem investigated in [Scarf, 1960] assumes that order quantity Q in each period can be as large as needed. In practice, one may want to impose the restriction that $0 \leq Q \leq B$, where B is a positive value denoting the maximum order quantity in each period.

We generalise $\widehat{C}_n(x)$ and $\widehat{G}_n(x)$ to reflect capacity restrictions

$$C_n(x) \triangleq \min_{x \leq y \leq x+B} \left\{ c(y-x) + L_n(y) + \int_0^\infty C_{n-1}(y-\xi) f_n(\xi) \mathrm{d}\xi \right\}; \tag{1}$$

$$G_n(y) \triangleq vy + L_n(y) + \int_0^\infty C_{n-1}(y-\xi) f_n(\xi) \mathrm{d}\xi. \tag{2}$$

Jacob Wijngaard. **An inventory problem with constrained order capacity**. Technical report, Department of Mathematics, Technological University Eindhoven, the Netherlands, 1972

Wijngaard conjectured that an optimal strategy may feature a so-called **modified (s, S) structure**:

> *"if the inventory level is greater or equal to s, do not order; otherwise, order up to S, or as close to S as possible, given the ordering capacity."*

Unfortunately, both [Wijngaard, 1972] and [Shaoxiang and Lambrecht, 1996] provided **counterexamples** that ruled out the optimality of a modified (s, S) policy.

Example 35. *Consider a planning horizon of* $n = 20$ *periods and a stationary demand* d *distributed in each period according to the following probability mass function:* $\Pr\{d = 6\} = 0.95$ *and* $\Pr\{d = 7\} = 0.05$*. Other problem parameters are* $K = 22$*,* $B = 9$*,* $h = 1$ *and* $p = 10$ *and* $v = 1$*; however, note that for a stationary problem with a sufficiently long horizon,* v *can be safely ignored. The authors also consider a discount factor* $\alpha = 0.9$*, which can be easily embedded in the code we presented.*

In Table 12 we report the tabulated optimal policy as illustrated in [Shaoxiang, 2004, p. 417]. It is easy to see that this policy does not follow a modified (s, S) rule as defined above.

Starting inventory level	-3	-2	-1	0	1	2	3	4	5	6	7
Optimal order quantity	9	8	7	9	8	7	9	8	7	0	0

Table 12 Optimal policy as illustrated in [Shaoxiang, 2004, p. 417].

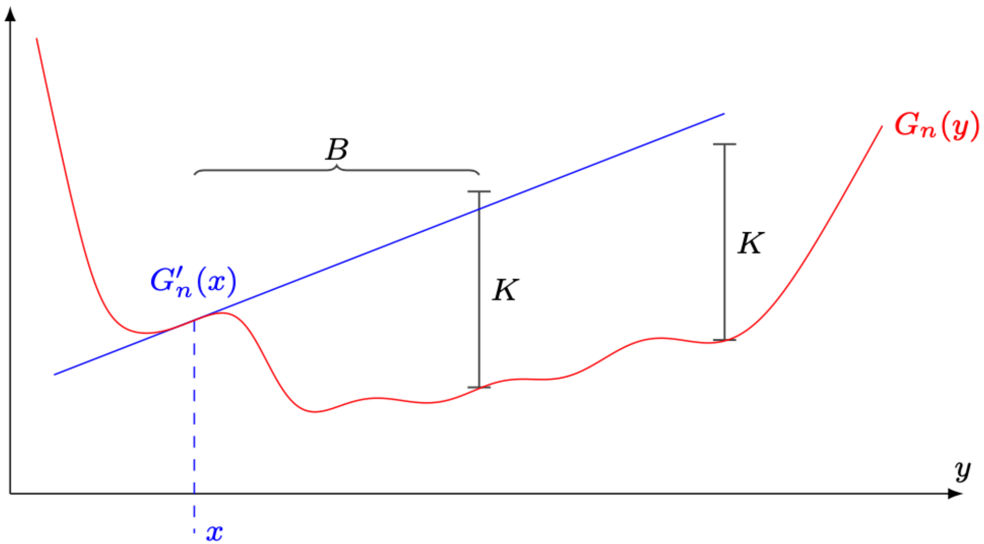
Guillermo Gallego and Alan Scheller-Wolf. **Capacitated inventory problems with fixed order costs: Some optimal policy structure**. European Journal of Operational Research, 126(3):603–613, 2000

We next introduce[3] "(K, B)-convexity 1" (KBC1) for a function g (Gallego & Scheller-Wolf, 2000).

Definition 3. *Let $K \geq 0$, $B \geq 0$, g is KBC1 if it satisfies*

$$(K + g(x + a) - g(x))/a \geq (g(y) - g(y - b))/b$$

for $0 < a \leq B$, $0 < b \leq B$, and $y \leq x$.



Chen Shaoxiang. **The infinite horizon periodic review problem with setup costs and capacity constraints: A partial characterization of the optimal policy**. Operations Research, 52, 409–421, 2004

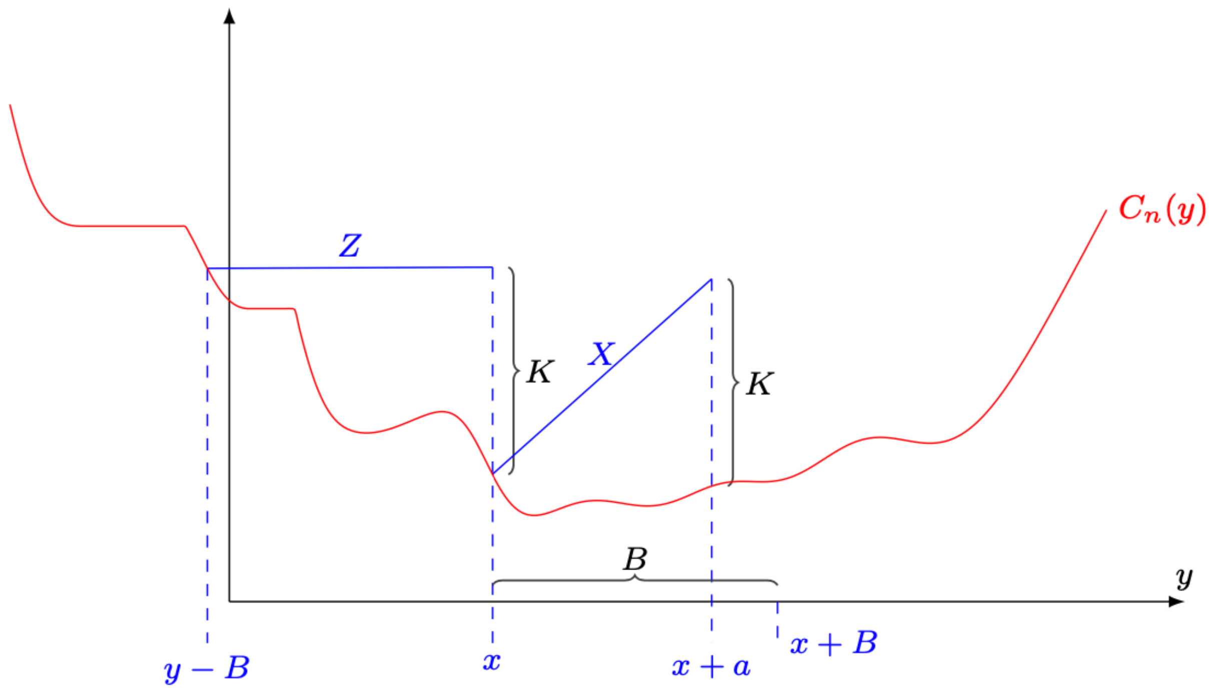
C. Shaoxiang & M. Lambrecht. **X-Y band and modified (s,S) policy**. Operations Research, 44, 1013–1019, 1996

We next introduce "(K, B)-convexity 2" (KBC2) for a function g (Shaoxiang, 2004).

Definition 4. *Let $K \geq 0$, $B \geq 0$, g is KBC2 if it satisfies*

$$(K + g(x + a) - g(x))/a \geq (K + g(y) - g(y - B))/B$$

for $0 < a \leq B$ and $y \leq x$.



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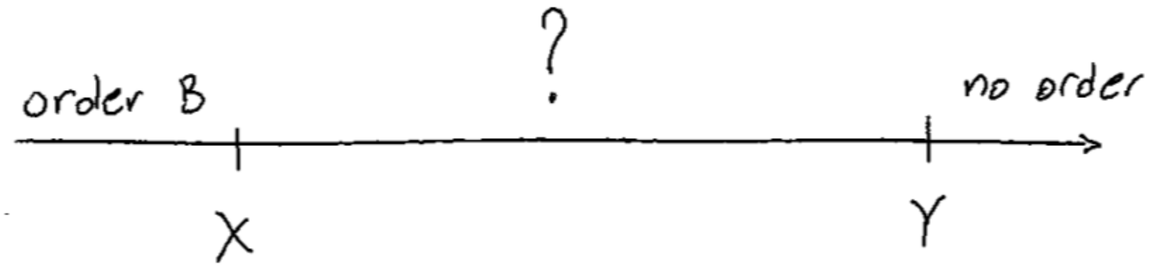
Originally in (Shaoxiang & Lambrecht, 1996), and then by introducing the concept of KBC2 in (Shaoxiang, 2004), Shaoxiang & Lambrecht established existence of a level $Y \triangleq s_m$ beyond which it is not optimal to order, and of another level $X \triangleq Y - B$ below which it is optimal to order up to capacity. The optimal policy therefore features a so-called "$X - Y$ band" structure.



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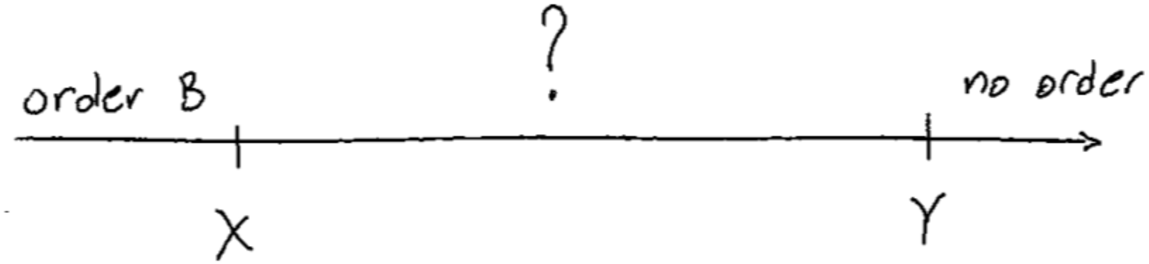

Gallego & Scheller-Wolf (2000) further characterised the structure of the optimal policy within Shaoxiang & Lambrecht's $X - Y$ band. In particular, they showed that

$$C_n(x) = \begin{cases} G_n^B(x) & x < \min\{s' - B, s\} \\ \alpha \min\{-vx + G_n(x), G_n^B(x)\} + (1-\alpha) G_n^S(x) & \min\{s' - B, s\} \leq x < \max\{s' - B, s\} \\ \min\{-vx + G_n(x), G_n^S(x)\} & \max\{s' - B, s\} \leq x \leq s' \\ -vx + G_n(x) & x > s' \end{cases} \quad (3)$$

where

$$G_n^B(x) \triangleq K - vx + G_n(x + B)$$

$$G_n^S(x) \triangleq K - vx + \min_{x \leq y \leq x+B} G_n(y)$$

$$s \triangleq \inf\{x | K + \min_{x \leq y \leq x+B} G_n(y) - G_n(x) \geq 0\}$$

$$s' \triangleq \max\{x \leq S_m | K + \min_{x \leq y \leq x+B} G_n(y) - G_n(x) \leq 0\}$$

and α is an indicator variable that takes value 1 if $s' - s > B$, and 0 otherwise.

where

$$
\begin{aligned}
&G_n^B(x) \triangleq K - vx + G_n(x+B)\\
&G_n^S(x) \triangleq K - vx + \min_{x\leq y\leq x+B} G_n(y)\\
&s \triangleq \inf\{x | K + \min_{x\leq y\leq x+B} G_n(y) - G_n(x) \geq 0\}\\
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\end{aligned}
$$

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Lemma 5. $s' - B < s \leq s'$

By leveraging Lemma 5, it is possible to further simplify Gallego & Scheller-Wolf's structure of the optimal policy as follows. To the best of our knowledge, this simplified policy structure has not been previously discussed in the literature.

Lemma 6.

$$
C_n(x) = \begin{cases} G_n^B(x) & x < s_m - B \\ G_n^S(x) & s_m - B \leq x < s \\ \min\{-vx + G_n(x), G_n^S(x)\} & s \leq x \leq s_m \\ -vx + G_n(x) & x > s_m \end{cases} \tag{4}
$$

Guillermo Gallego and Alan Scheller-Wolf. **Capacitated inventory problems with fixed order costs: Some optimal policy structure**. *European Journal of Operational Research*, 126(3):603–613, 2000

The continuous order property in Definition 5 has been originally conjectured by Gallego & Scheller-Wolf (2000), and it was later further investigated by Chan & Song (2003). Gallego & Scheller-Wolf (2000) wrote:

> A number of problems still remain. The most vexing is the possibility that under the current structure there could exist a number of intervals [...] where it is optimal to start and stop ordering. An optimal policy with a single continuous interval over which ordering is prescribed, as was found for all of the cases tested [...], is much more analytically appealing. [...] Unfortunately, the proof of this has thus far eluded us. It should be mentioned that it is likewise possible, although we believe it unlikely, that such a structure simply does not exist. To show this requires a problem instance in which the optimal policy has multiple disjoint intervals in which ordering is optimal. Our computational study suggests that this is not the case.

G. H. Chan & Y. Song. **A dynamic analysis of the single-item periodic stochastic inventory system with order capacity**. *European Journal of Operational Research*, 146, 529-542, 2003

Chan & Song (2003) wrote:

> If our conjecture [the continuous order property] holds, the computational time for obtaining the optimal ordering policy parameters can be further reduced [...]. We can only show that this conjecture holds for a special case where [the capacity] is large enough [...]. It should be an interesting problem for researchers to prove or disprove the conjecture is true for small [capacity].

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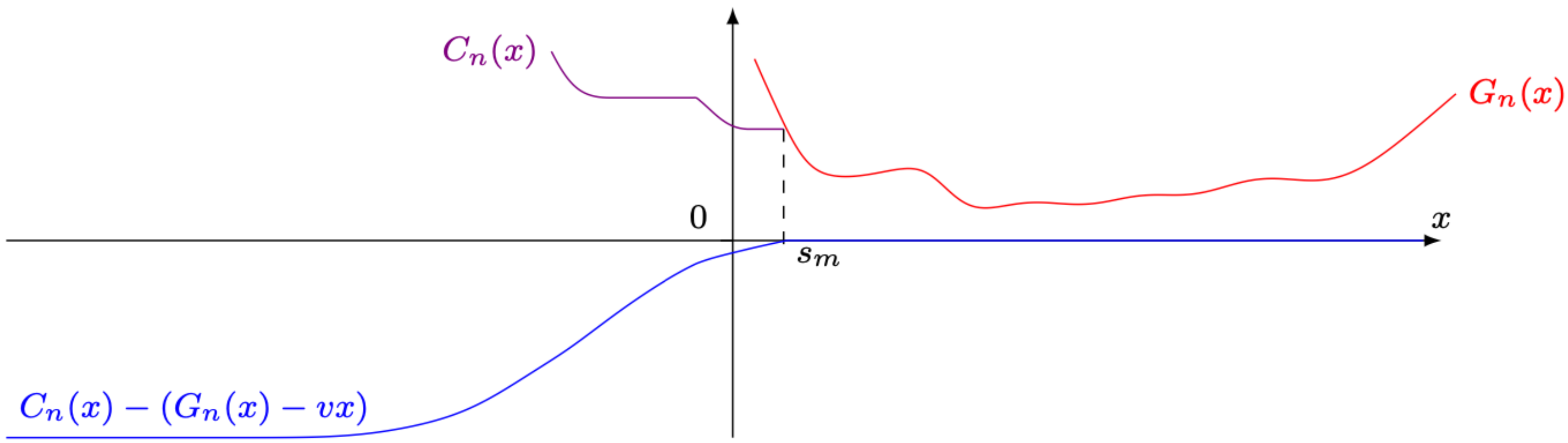
Definition 6 (Continuous Order Property). *Let x_0 be an inventory level at which it is optimal to place an order, C_n is said to have the continuous order property if it is optimal to place an order at y, for all $y < x_0$.*

Lemma 7. *If C_n has the continuous order property, $\{x|C_n(x) - (G_n(x) - vx) < 0\}$ is a convex set.*



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Consider C_n as defined in Eq (1), let this function be (K, B)-convex, and assume that the continuous order property holds. When inventory falls below the reorder threshold s_m, defined in Lemma 3, the optimal policy takes the following form: at the beginning of each period, let x be the initial inventory, the order quantity Q is computed as

$$Q = \begin{cases} \min\{S_k - x, B\} & s_{k-1} < x \leq s_k, \\ 0 & x > s_m; \end{cases} \tag{5}$$

where $k = 1, \ldots, m$ and $s_0 = -\infty$. In essence, the policy features m reorder thresholds $s_1 < s_2 < \ldots < s_m$ and associated order-up-to-levels $S_1 < S_2 < \ldots < S_m$; at the beginning of each period, if inventory drops between reorder threshold s_k and reorder threshold s_{k-1}, it is optimal to order a quantity $Q = \min\{S_i - x, B\}$. For convenience, we denote the case $Q = B$ as *saturated ordering*, and the case $0 < Q < B$ as *unsaturated ordering*. We shall name this control rule *modified multi-(s, S) policy*, or (s_k, S_k) policy in short. This policy structure was also described in (Gallego & Scheller-Wolf, 2000, p. 612); however, Gallego & Scheller-Wolf did not establish a relation between the continuous order property and the optimality of this policy.

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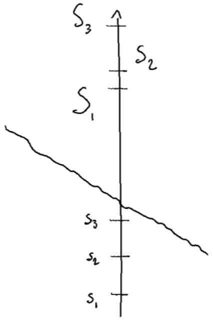

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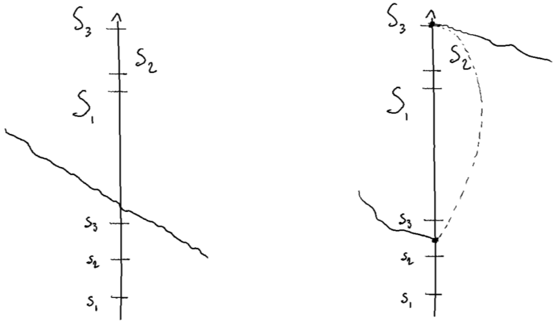

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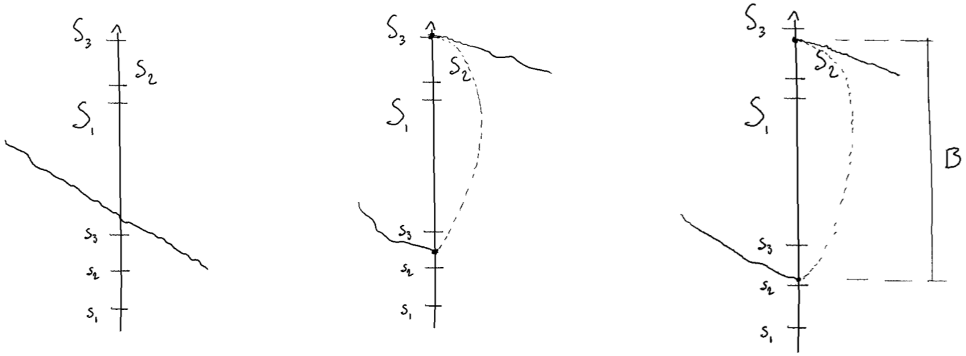

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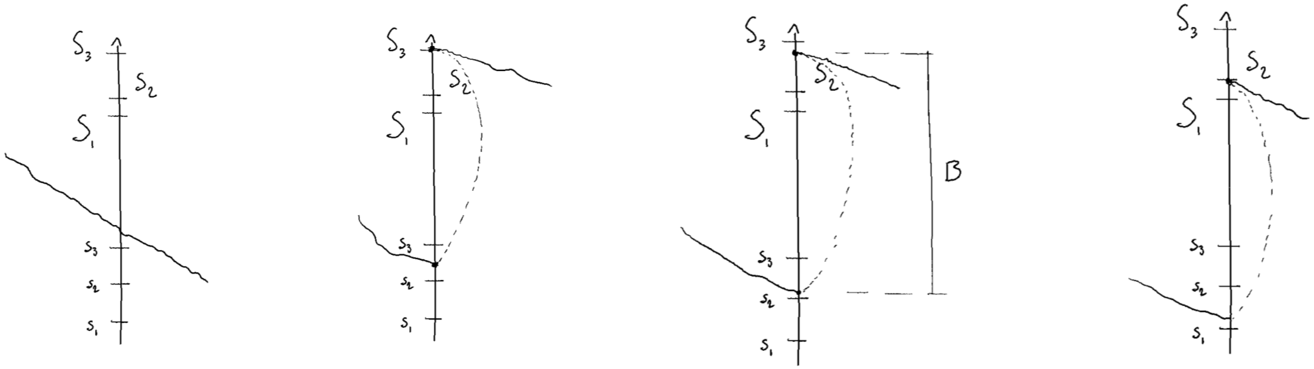

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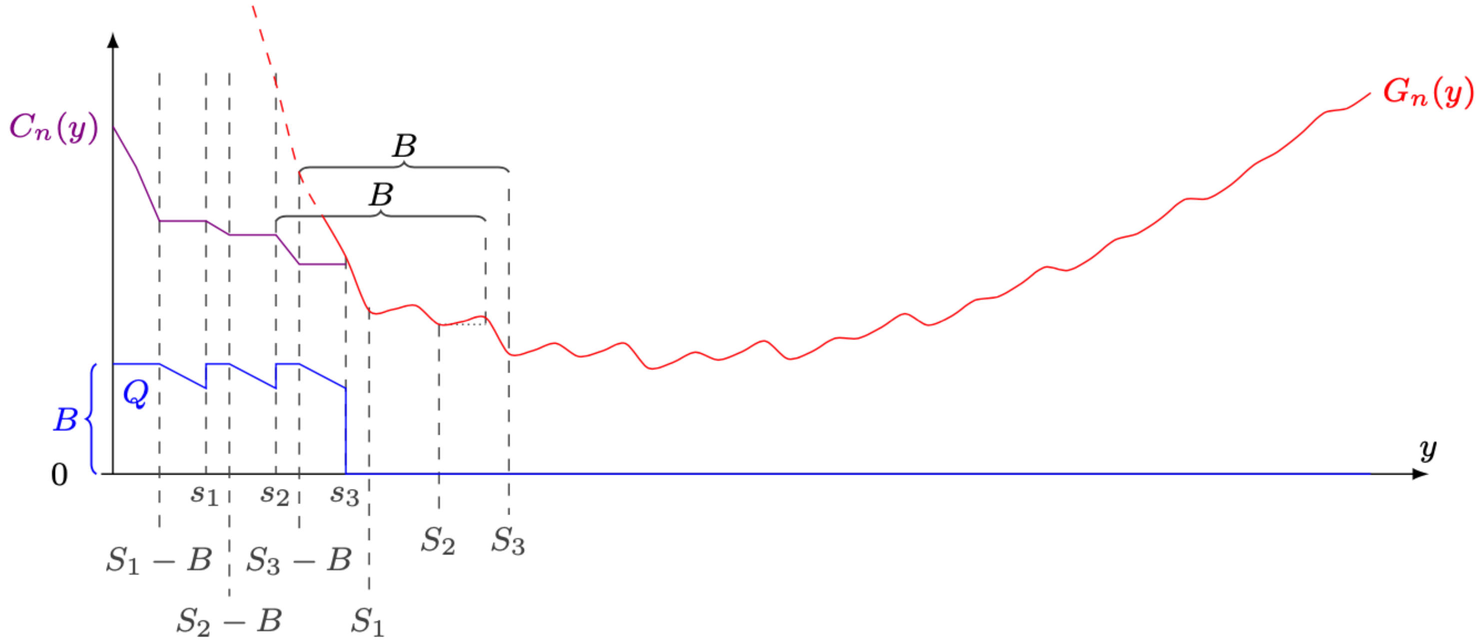
We hereby illustrate that an (s_k, S_k) ordering policy is optimal for the numerical example originally presented in (Shaoxiang & Lambrecht, 1996, p. 1015) and also investigated in (Shaoxiang, 2004) under an infinite horizon.

Example 5. *Consider a planning horizon of* $n = 20$ *periods and a stationary demand* d *distributed in each period according to the following probability mass function:* $\Pr\{d = 6\} = 0.95$ *and* $\Pr\{d = 7\} = 0.05$. *Other problem parameters are* $K = 22$, $B = 9$, $h = 1$ *and* $p = 10$ *and* $v = 1$*; note that, if the planning horizon is sufficiently long,* v *can be safely ignored. The discount factor is* $\alpha = 0.9$.

Starting inventory level	-3	-2	-1	0	1	2	3	4	5	6	7
Optimal order quantity	9	8	7	9	8	7	9	8	7	0	0

Table C.11: Optimal policy as illustrated in (Shaoxiang, 2004, p. 417)



s_k	S_k
-1	6
2	9
5	12

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Corollary 2. *If the continuous order property holds, the* (s_k, S_k) *policy generalises the* $X - Y$ *band discussed in (Shaoxiang, 2004).*

Corollary 3. *If the continuous order property holds, the* (s_k, S_k) *policy generalises the policy discussed in (Gallego & Scheller-Wolf, 2000).*

Corollary 4. *If the continuous order property holds, the* (s_k, S_k) *policy generalises the* (s, S) *policy discussed in (Scarf, 1960).*

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d_1	34 (0.018)	159 (0.888)	281 (0.046)	286 (0.048)
d_2	14 (0.028)	223 (0.271)	225 (0.170)	232 (0.531)
d_3	5 (0.041)	64 (0.027)	115 (0.889)	171 (0.043)
d_4	35 (0.069)	48 (0.008)	145 (0.019)	210 (0.904)

Table 2: Probability mass functions of the nonstationary demand d_t considered in Example 3.

Example 3. *Consider a planning horizon of $n = 4$ periods and a nonstationary demand d_t distributed in each period t according to the probability mass function shown in Table 2. Other problem parameters are $K = 250$, $B = 41$, $h = 1$ and $p = 26$ and $v = 0$.*

In Table 3 we report an extract of the tabulated optimal policy in which the continuous order property is violated (Fig. 7).

Starting inventory level	593	594	595	596	597	598	599	600	601
Optimal order quantity	41	40	39	38	37	36	35	34	33
Starting inventory level	602	603	604	605	606	607	608	609	610
Optimal order quantity	0	0	0	0	0	0	0	0	0
Starting inventory level	611	612	613	614	615	616	617	618	619
Optimal order quantity	0	0	0	0	0	41	41	41	0

Table 3

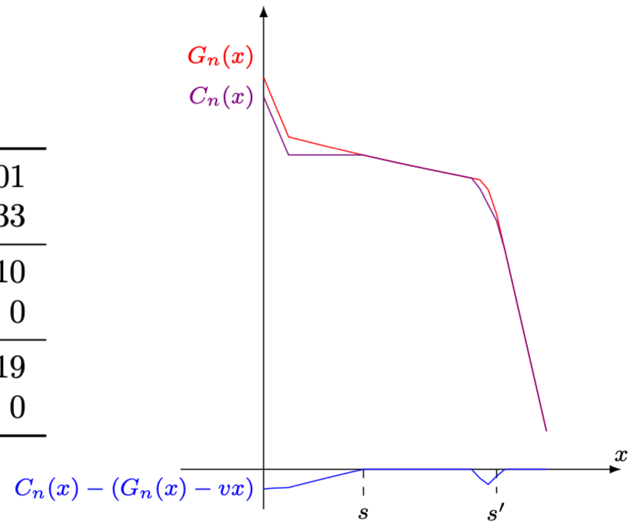

Figure 7

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To prove that $\{x|C_n(x) - (G_n(x) - vx) < 0\}$ is a convex set, it is sufficient to show that the function

$$V_n(x) \triangleq C_n(x) - (G_n(x) - vx)$$

is nondecreasing in x for each n. Let $[x]^- \triangleq \min\{0, x\}$, and note that

$$V_n(x) = [K + \min_{x \le y \le x+B} G_n(y) - G_n(x)]^-.$$

One may want to try and show by induction that $V_n(x)$ is nondecreasing in x for each n.

First, observe that

$$\begin{aligned} V_{n+1}(x) \quad &= [K + \min_{x \le y \le x+B} \quad (vy + L_{n+1}(y) + \textstyle\int_0^\infty C_n(y-\xi) f_{n+1}(\xi) \mathrm{d}\xi) \\ &\quad -(vx + L_{n+1}(x) + \textstyle\int_0^\infty C_n(x-\xi) f_{n+1}(\xi) \mathrm{d}\xi)]^-. \end{aligned}$$

To investigate whether $V_{n+1}(x)$ is nondecreasing, we shall analyse

$$K + \min_{x \le y \le x+B} v(y-x) + C_n(y) - C_n(x)$$

Gallego, G., & Toktay, L. B. (2004). **All-or-nothing ordering under a capacity constraint.** Operations Research, 52, 1001–1002.

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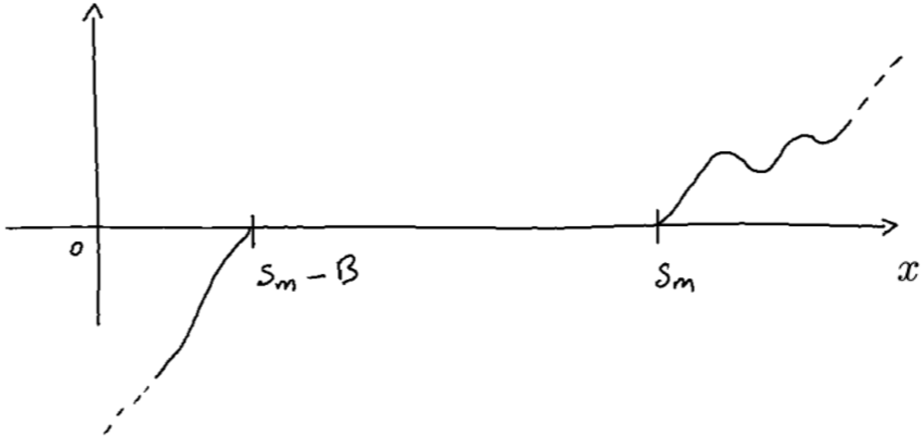

$$\left[K + \min_{x \leq y \leq x+B} v(y-x) + C_n(y) - C_n(x)\right]^- = \begin{cases} V_n(x+B) & x \leq s_m - B \\ 0 & s_m - B < x \leq s_m \\ 0 & x > s_m \end{cases}$$

is nondecreasing. However, it is not possible to determine if $[K + \min_{x \leq y \leq x+B} v(y-x) + \int_0^\infty (C_n(y-\xi) - C_n(x-\xi)) f_{n+1}(\xi) \mathrm{d}\xi]^-$ is nondecreasing; and reintroducing term $\min_{x \leq y \leq x+B} L_{n+1}(y) - L_{n+1}(x)$ only worsen the matter. But because of the behavior of $[K + \min_{x \leq y \leq x+B} v(y-x) + C_n(y) - C_n(x)]^-$ in intervals $s_m - B < x \leq s_m$ and $x \leq s_m - B$, one may observe that a $V_{n+1}(x)$ function featuring some decreasing regions may be produced by the convolution $\int_0^\infty (C_n(y-\xi) - C_n(x-\xi)) f_{n+1}(\xi) \mathrm{d}\xi$, provided demand is sufficiently "lumpy." In other words, the instance must feature demand whose probability mass function features some values larger than B possessing non negligible probability mass. A demand that is so structured may ensure that the convolution "bends" sufficiently $V_{n+1}(x)$ beyond s_m so that it turns negative.

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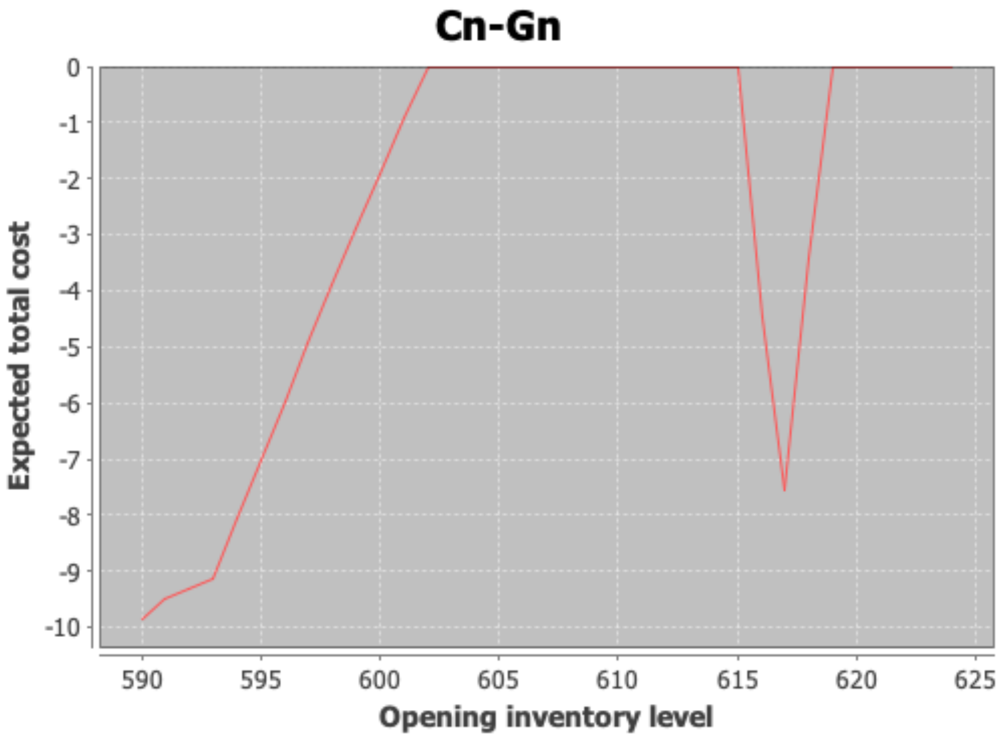

On the basis of this observation, we have generated several random instances as follows. The fixed ordering cost is a randomly generated value uniformly distributed between 1 and 500; holding cost is 1; penalty cost is a randomly generated value uniformly distributed between 1 and 30; the ordering capacity is a randomly generated value uniformly distributed between 20 and 200; demand distribution in each period is obtained as follows: the probability mass function comprises only four values in the support, one of these values must fall below the given order capacity, the other three values must fall above, and be smaller or equal to 300; probability masses are then allocated uniformly to each of these values. The Java code to generate instances that violate the continuous order property is available on `http://gwr3n.github.io/jsdp/`.[5]

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Albeit in the previous section we demonstrated that it is possible to construct instances for which the continuous order property does not hold, we must underscore that these instances are hard to generate, as they do not show up in numerical experiments featuring conventional parameter ranges found in the literature. This is also the reason why the conjecture in (Gallego & Scheller-Wolf, 2000; Chan & Song, 2003) remained open for over twenty years.

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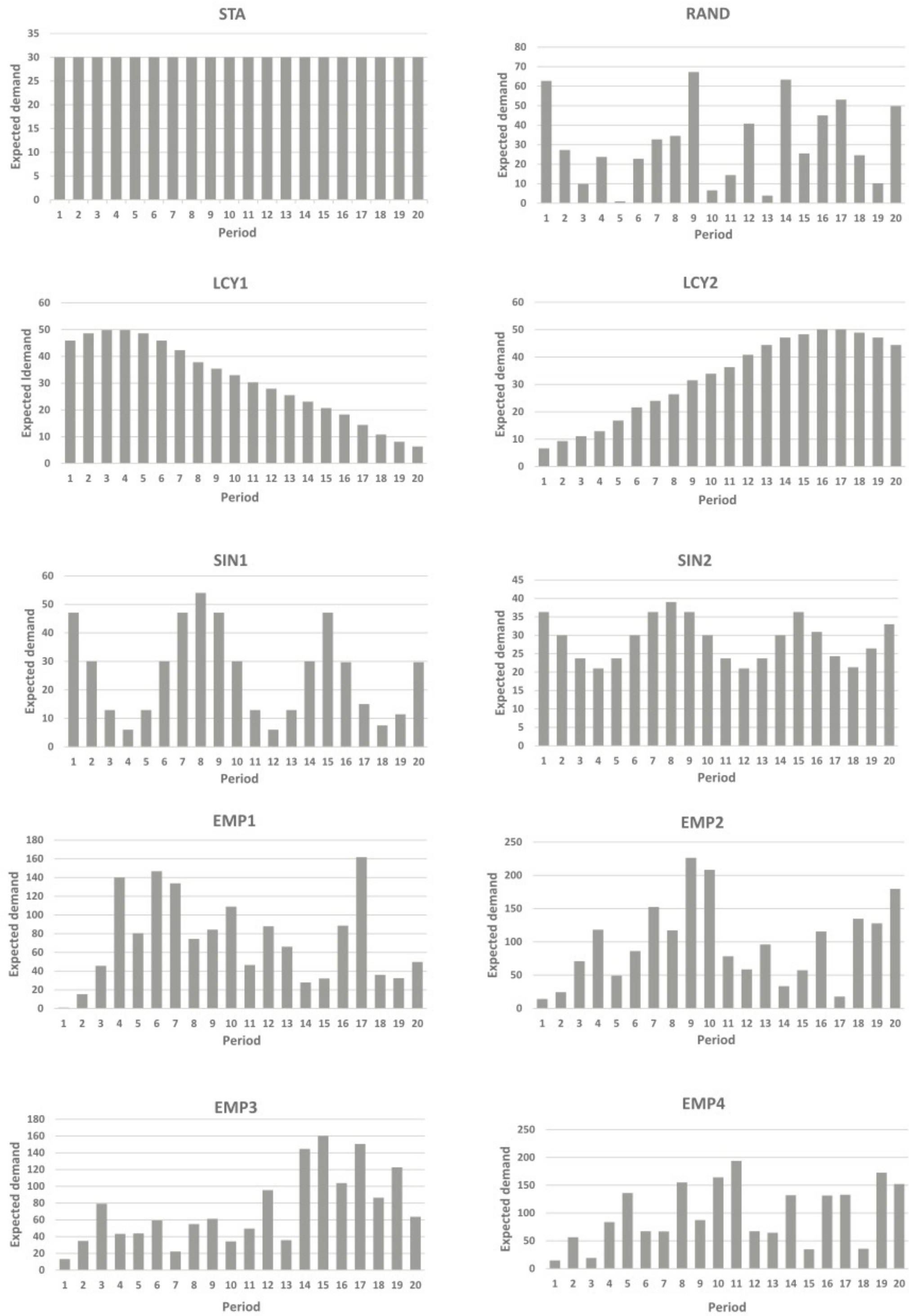
Since we adopt a full factorial design, we consider 810 instances for discrete uniform, geometric, and Poisson distributed demand, respectively; and 2430 instances for normal, lognormal, and gamma distributed demands, respectively, since in these latter cases we must also consider the three levels of the coefficient of variation. In total, our computational study comprises 9720 instances. Our experimental design is similar to that investigated in a number of existing studies (see e.g. Xiang et al., 2018; Dural-Selcuk et al., 2020).


STA
Expected demand
Period
RAND
Expected demand
Period
LCY1
Expected ldemand
Period
LCY2
Expected demand
Period
SIN1
Expected demand
Period
SIN2
Expected demand
Period
EMP1
Expected demand
Period
EMP2
Expected demand
Period
EMP3
Expected demand
Period
EMP4
Expected demand
Period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		modified (s, S) % optimality gap		modified multi-(s, S) max thresholds	instances
		avg	max		
K	250	0.101	1.930	5	810
	500	0.068	1.424	6	810
	1000	0.029	0.570	6	810
v	2	0.094	1.930	6	810
	5	0.062	0.923	6	810
	10	0.043	0.731	6	810
p	5	0.046	0.894	5	810
	10	0.071	1.585	6	810
	15	0.081	1.930	6	810
B	2.0D	0.062	1.930	5	810
	3.0D	0.080	1.585	5	810
	4.0D	0.057	1.424	6	810
Demand	EMP1	0.125	1.585	4	243
	EMP2	0.095	1.930	4	243
	EMP3	0.100	1.424	6	243
	EMP4	0.130	1.312	5	243
	LC1	0.029	0.308	5	243
	LC2	0.032	0.894	6	243
	RAND	0.052	1.290	5	243
	SIN1	0.036	0.421	5	243
	SIN2	0.037	0.708	5	243
	STA	0.027	0.317	5	243
c_v	0.1	0.109	1.930	6	810
	0.2	0.050	1.013	5	810
	0.3	0.040	0.570	4	810
Overall		0.066	1.930	6	2430

Table 10: Pivot table for our computational study: gamma demand.

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Concluding remarks

The periodic review single-item single-stocking location stochastic inventory system under nonstationary demand, complete backorders, a fixed ordering cost component, and order quantity capacity constraints is one of the fundamental problems in inventory management.

A long standing open question in the literature is whether a policy with a single continuous interval over which ordering is prescribed is optimal for this problem. The so-called "continuous order property" conjecture was originally posited by Gallego & Scheller-Wolf (2000), and later also investigated by Chan & Song (2003). To the best of our knowledge, to date this conjecture has never been confirmed or disproved.

In this work, we provided a numerical counterexample that violates the continuous order property. This closes a fundamental and long standing problem in the literature: a policy with a single continuous interval over which ordering is prescribed is not optimal.

Gallego & Scheller-Wolf (2000) provided a partial characterisation of the optimal policy to the problem. In light of the results presented in (Shaoxiang, 2004), we showed how to simplify the optimal policy structure presented by Gallego & Scheller-Wolf (2000). Gallego & Scheller-Wolf (2000) also briefly sketched the form that an optimal policy would take under moderate values of K. We formalised this discussion and provided a full characterisation of the optimal policy for instances for which the continuous order property holds. In particular, we showed that under this assumption the optimal policy takes the *modified multi-(s,S) form.*

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Concluding remarks

By leveraging an extensive computational study, we showed that instances violating the continuous order property do not surface when realistic cost configurations and demand patterns investigated in the literature are considered. The modified multi-(s,S) ordering policy can therefore be considered near-optimal for the problem under scrutiny. Moreover, we observed that the number of thresholds in a modified multi-(s,S) policy remains less or equal to 6 in each period. Finally, we showed that a well-known heuristic policy, the modified (s,S) policy (Wijngaard, 1972), also performs well across all instances considered.

Since a policy with a single continuous interval over which ordering is prescribed is not optimal in general, future works may focus on establishing what restrictions (if any) to the problem statement, e.g. nature of the demand distribution, may ensure that such a policy is optimal.

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Future works

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