Confidence-based optimization for the Newsvendor problem

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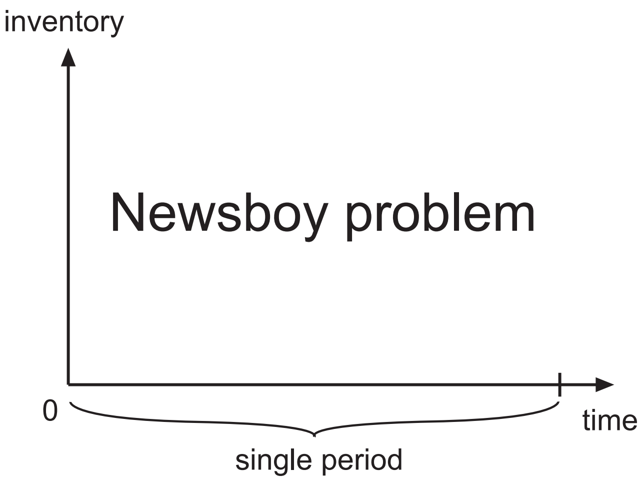
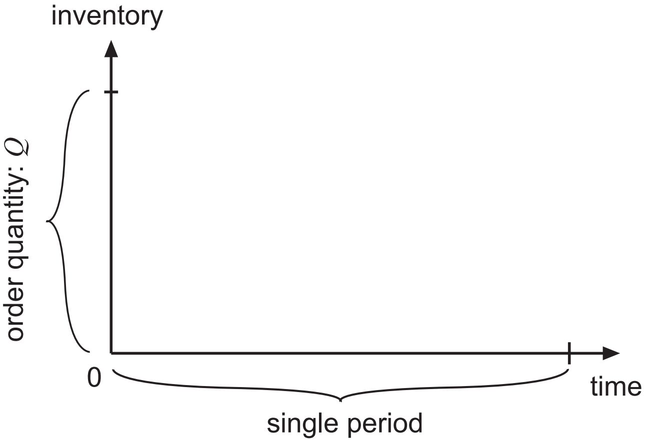
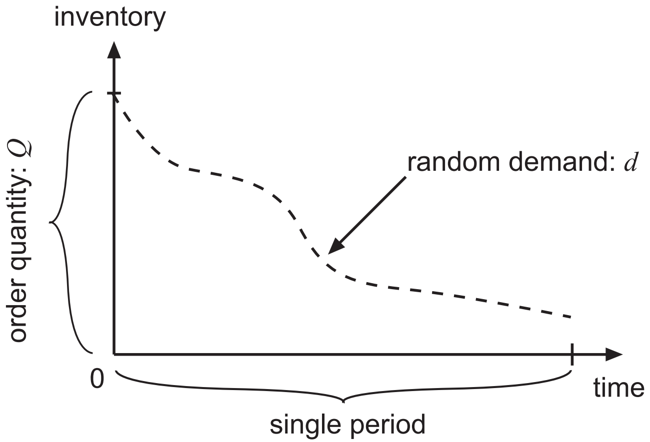
The Newsboy problem



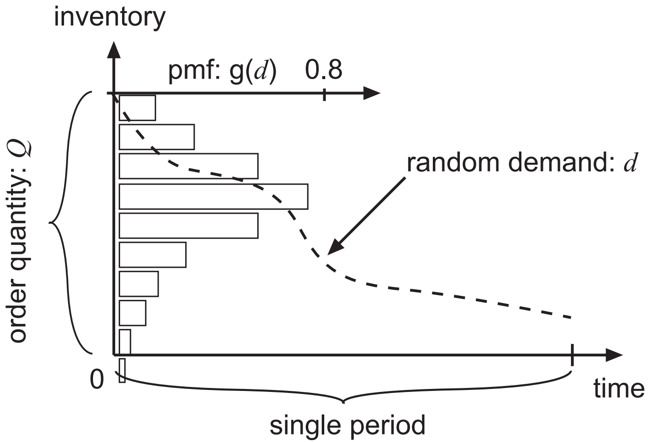
Order quantity



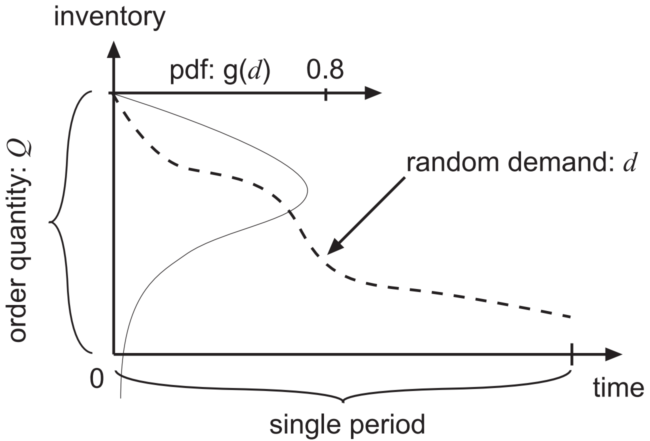
Demand structure



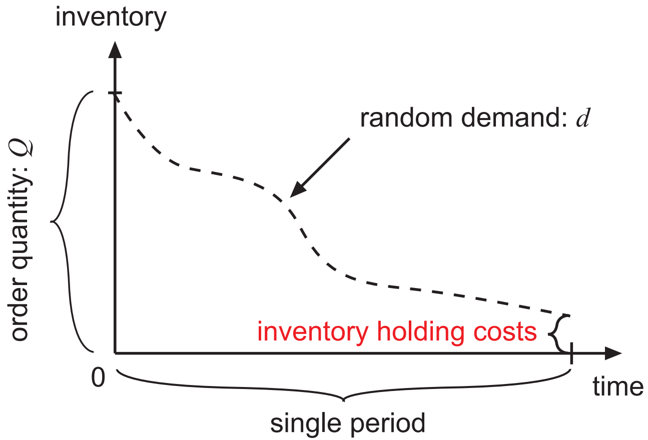
Demand structure



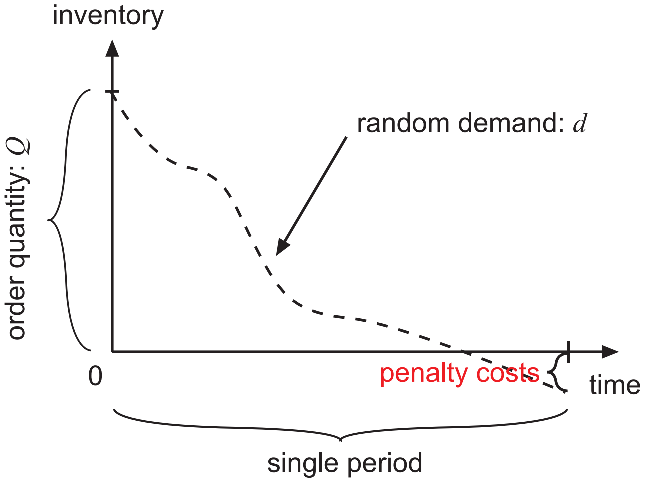
Demand structure



Cost structure



Cost structure



Mathematical formulation

Consider

- d: a **one-period** random demand that follows a **probability distribution** $f(d)$
- h: unit **holding cost**
- p: unit **penalty cost**

Let

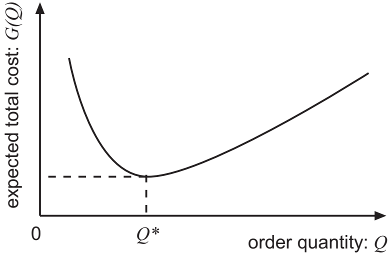
$$g(x) = hx^{+} + px^{-},$$

where $x^{+} = \max(x, 0)$ and $x^{-} = -\min(x, 0)$.

The **expected total cost** is $G(Q) = E[g(Q - d)]$, where $E[\cdot]$ denotes the expected value.

Solution method

If d is continuous, $G(Q)$ is **convex**.



The optimal order quantity is

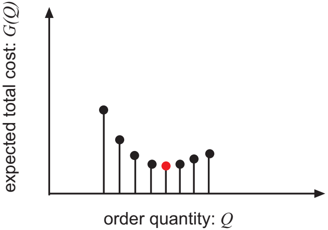
$$Q^* = \inf\{Q \geq 0 : \Pr\{d \leq Q\} = \frac{p}{p+h}\}.$$

Solution method

If d is discrete (e.g. Poisson),

$$\Delta G(Q) = h - (h+p)\,\Pr\{d > j\}$$

is **non-decreasing** in Q.



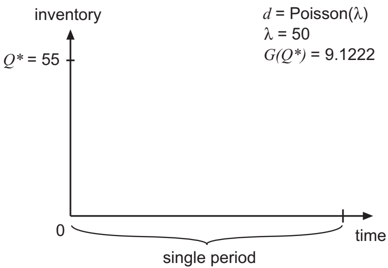
$$Q^* = \min\{Q \in \mathbb{N}_0 : \Delta G(Q) \geq 0\}.$$

Solution method: example

Demand follows a Poisson distribution *Poisson*(λ), with demand rate $\lambda = 50$.

Holding cost $h = 1$, penalty cost $p = 3$.

The optimal order quantity Q^* is equal to 55 and provides a cost equal to 9.1222.



Unknown distribution parameter(s)

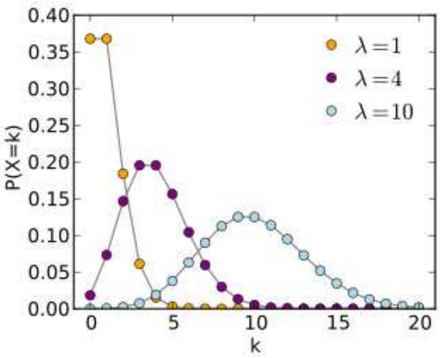
Assume now that the **demand distribution** is known, but one or more **distribution parameters** are unknown.

The decision maker has access to a set of M **past realizations of the demand**.

From these she has to estimate the **optimal order quantity** (or quantities) and the **associated cost**.

Unknown distribution parameter(s)

Poisson demand, probability mass function:



λ has to be **estimated** from past realizations.

Point estimates of the parameter(s)

Point estimates of the unknown parameters may be obtained from the available samples by using:

- **maximum likelihood estimators**, or
- the **method of moments**.

Point estimates for the parameters are then used **in place** of the unknown demand distribution parameters to compute:

- the estimated **optimal order quantity** $\widehat{Q}^*$, and
- the associated estimated **expected total cost** $G(\widehat{Q}^*)$.

Point estimates: example

M observed **past demand data** $d_1, \ldots, d_M$.

Demand follows a **Poisson distribution** $Poisson(\lambda)$, with demand rate λ.

We estimate λ using the **maximum likelihood estimator** (sample mean):

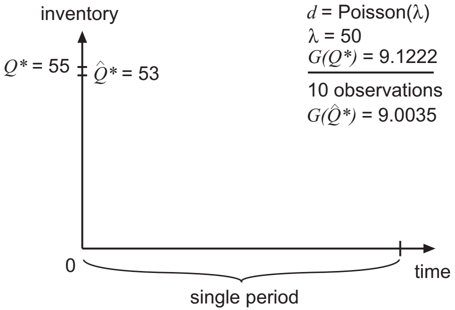
$$\widehat{\lambda} = \frac{1}{M} \sum_{i=1}^{M} \lambda_i .$$

The decision maker employs the distribution $Poisson(\widehat{\lambda})$ **in place** of the actual unknown demand distribution.

Point estimates: example

Holding cost: $h = 1$; penalty cost: $p = 3$;
observed past demand data:
$\{51, 54, 50, 45, 52, 39, 52, 54, 50, 40\}$.

$\widehat{\lambda} = 48.7$, $\widehat{Q}^* = 53$ and $G(\widehat{Q}^*) = 9.0035$.



Bayesian approach

The bayesian approach **infers** the distribution of parameter λ given some past observations d by applying **Bayes' theorem** as follows

$$p(\lambda|d) = \frac{p(d|\lambda)p(\lambda)}{\int p(d|\lambda)p(\lambda)\mathrm{d}\lambda}$$

where

$p(\lambda)$ is the **prior distribution** of λ, and

$p(\lambda|d)$ is the **posterior distribution** of λ given the observed data d.

Bayesian approach

The **prior distribution** describes **an estimate** of the likely values that the parameter λ might take, without taking the data into account. It is based on **subjective assessment** and/or **collateral data**.

A number of methods for constructing "**non-informative priors**" have been proposed (i.e. maximum entropy). These are meant to reflect the fact that the decision maker **ignores** of the prior distribution.

If prior and posterior distributions are in the same family, then they are called **conjugate distributions**.

Bayesian approach [Hill, 1997]

Hill [EJOR, 1997] proposes a **bayesian approach to the Newsvendor problem**.

He considers a number of distributions (Binomial, Poisson and Exponential) and **derives posterior distributions for the demand** from a set of given data.

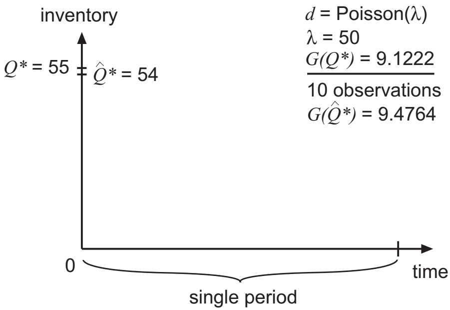
He adoptes **uninformative priors** to express an initial state of **complete ignorance** of the likely values that the parameter might take.

By using the posterior distribution he obtaines an **estimated optimal order quantity** and the respective **estimated expected total cost**.

Bayesian approach: example

Holding cost: $h = 1$; penalty cost: $p = 3$; observed past demand data: $\{51, 54, 50, 45, 52, 39, 52, 54, 50, 40\}$.

$\widehat{Q}^* = 54$ and $G(\widehat{Q}^*) = 9.4764$.



Drawbacks of existing approaches

Only provide point estimates of the order quantity and of the expected total cost.

Do not quantify the uncertainty associated with this estimate.

- How do we distinguish a case in which we only have 10 past observations vs a case with 1000 past observations?

The bayesian approach produces results that are "**biased**" by the selection of the prior; the posterior distribution **may not satisfy** Kolmogorov axioms.

J. Neyman. Outline of a theory of statistical estimation based on the classical theory of probability. Philosophical Transactions of the Royal Society of London, 236:333—380, 1937

An alternative approach

We propose a solution method based on **confidence interval analysis** [Neyman, 1937].

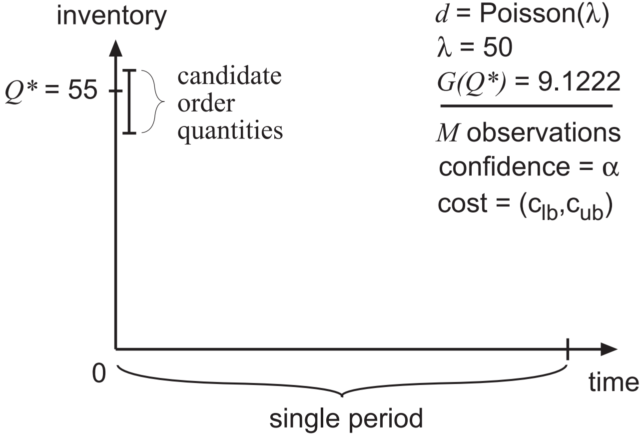
Observation

Since we operate under partial information, it may not be possible to uniquely determine “the” optimal order quantity and the associated exact cost.

We argue that a possible approach consists in **determining a range** of “candidate” optimal order quantities and **upper and lower** bounds for the **cost** associated with these quantities.

This range will contain the real optimum according to a **prescribed confidence probability** α.

An alternative approach



d = Poisson(λ)

$\lambda = 50$

$G(Q^*) = 9.1222$

M observations

confidence = α

cost = (c_{lb}, c_{ub})

Confidence interval for λ

Consider a set of M samples d_i drawn from a random demand d that is distributed according to a Poisson law with unknown parameter λ.

We construct a **confidence interval** for the unknown demand rate λ as follows

$$\lambda_{lb} = \min\{\lambda | \Pr\{Poisson(M\lambda) \geq \bar{d}\} \geq (1-\alpha)/2\},$$
$$\lambda_{ub} = \max\{\lambda | \Pr\{Poisson(M\lambda) \leq \bar{d}\} \geq (1-\alpha)/2\},$$

where $\bar{d} = \sum_{i=0}^{M} d_i$.

A **closed form expression** for this interval has been proposed by Garwood [1936] based on the chi-square distribution.

Confidence interval for λ: example

Consider the set of 10 samples

$$\{51, 54, 50, 45, 52, 39, 52, 54, 50, 40\},$$

and $\alpha = 0.9$.

The confidence interval for the unknown demand rate λ is

$$(\lambda_{lb}, \lambda_{ub}) = (45.1279, 52.4896),$$

Note that, by chance, this interval covers the actual demand rate $\lambda = 50$ used to generate the samples.

Candidate order quantities

Let Q^*_{lb} be the **optimal order quantity** for the Newsvendor problem under a $Poisson(\lambda_{lb})$ demand.

Let Q^*_{ub} be the **optimal order quantity** for the Newsvendor problem under a $Poisson(\lambda_{ub})$ demand.

Since $\Delta G(Q)$ is **non-decreasing** in Q, according to the available information, **with confidence probability** α, the optimal order quantity Q^* is a **member** of the set $\{Q^*_{lb}, \ldots, Q^*_{ub}\}$.

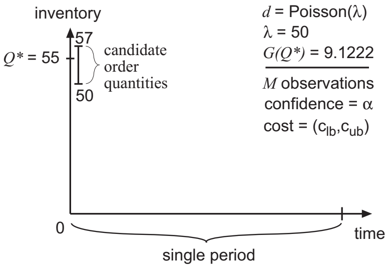
Candidate order quantities: example

Consider the set of 10 samples

$$\{51, 54, 50, 45, 52, 39, 52, 54, 50, 40\},$$

and $\alpha = 0.9$.

The candidate order quantities are



Confidence interval for the expected total cost

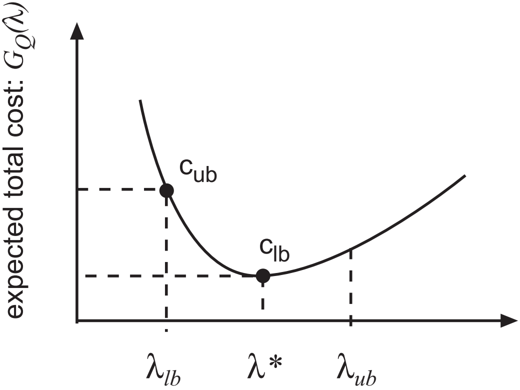
For a **given order quantity** Q we can prove that

$$G_Q(\lambda) = \begin{aligned} & h \sum_{i=0}^{Q} \Pr\{Poisson(\lambda) = i\}(Q - i) + \\ & p \sum_{i=Q}^{\infty} \Pr\{Poisson(\lambda) = i\}(i - Q), \end{aligned}$$

is **convex** in λ.

Upper (c_{ub}) and **lower** (c_{lb}) **bounds** for the cost associated with a solution that sets the order quantity to **a value in the set** $\{Q^*_{lb}, \ldots, Q^*_{ub}\}$ can be easily obtained by using **convex optimization** approaches to find the λ^* that maximizes or minimizes this function over $(\lambda_{lb}, \lambda_{ub})$.

Confidence interval for the expected total cost



for $Q \in \{Q^*_{lb}, \ldots, Q^*_{ub}\}$.

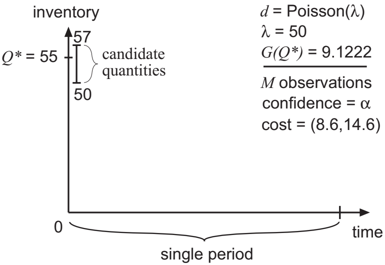
Expected total cost: example

Consider the set of 10 samples

$$\{51, 54, 50, 45, 52, 39, 52, 54, 50, 40\},$$

and $\alpha = 0.9$.

The upper and lower bound for the expected total cost are



Expected total cost: example

Assume we decide to order 53 items, according to what a **MLE approach** suggests.

As we have seen, **MLE estimates** an expected total cost of 9.0035 (note that the real cost we would face is 9.3693).

If we compute $c_{lb} = 8.9463$ and $c_{ub} = 11.0800$, then we know that with $\alpha = 0.9$ confidence this interval **covers the real cost** we are going to face by ordering 53 units.

Similarly, the Bayesian approach **only prescribes** $\widehat{Q}^* = 54$ and estimates $G(\widehat{Q}^*) = 9.4764$ (real cost is 9.1530), while we know that $c_{lb} = 9.0334$ and $c_{ub} = 10.3374$.

Lost sales

Consider the case in which **unobserved lost sales** occurred and the M observed past demand data, $d_1, \ldots, d_M$, **only reflect** the number of customers that purchased an item **when the inventory was positive**.

The analysis discussed above **can still be applied** provided that the confidence interval for the unknown parameter λ of the *Poisson*(λ) demand is computed as

$$\lambda_{lb} = \min\{\lambda | \Pr\{Poisson(\widehat{M}\lambda) \geq \bar{d}\} \geq (1-\alpha)/2\},$$
$$\lambda_{ub} = \max\{\lambda | \Pr\{Poisson(\widehat{M}\lambda) \leq \bar{d}\} \geq (1-\alpha)/2\}.$$

where $\widehat{M} = \sum_{j=1}^{M} T_j$, and $T_j \in (0,1)$ denotes **the fraction of time** in day j — for which a demand sample d_j is available — during which the **inventory was positive**.

Binomial demand

N customer enter the shop on a given day, the **unknown purchase probability** of the Binomial demand is $q \in (0, 1)$.

The analysis is **similar** to that developed for a Poisson demand.

Also in this case we prove that $G_Q(q)$ is **convex** in q.

Lost sales can be **easily incorporated** in the analysis.

Exponential demand

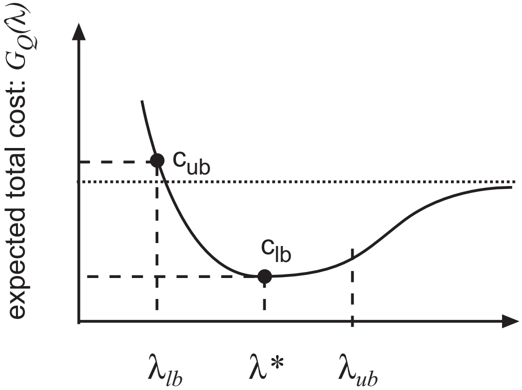
The interval of candidate order quantities can be **easily identified**.

The analysis on the expected total cost is **complicated** by the fact that $G_Q(\lambda)$ is **not convex**.

Extension to include lost sales is **difficult**.

Exponential demand

A **number of properties** of $G_Q(\lambda)$ can be exploited to find **upper and lower bounds for the expected total cost**.



Exponential demand

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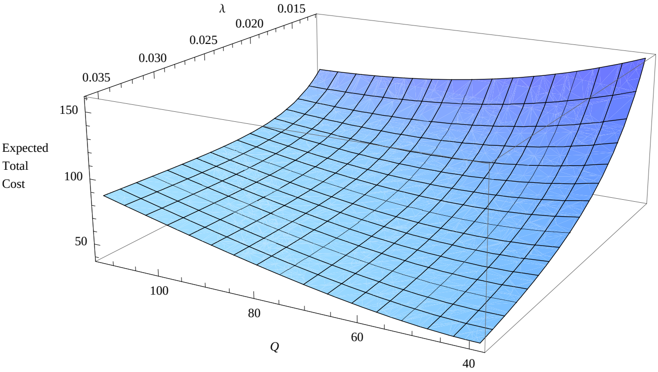

Discussion

We presented a **confidence-based optimization** strategy to the Newsboy problem with **unknown demand distribution parameter(s)**.

We applied our approach to three **maximum entropy** probability distributions of the **exponential family**.

We showed the **advantages of our approach** over two existing strategies in the literature.

For two demand distributions we extended the analysis to include **lost sales**.

Future works

Consider **other probability distributions** (e.g. Normal, LogNormal etc.).

Further **develop the analysis on lost sales** for the Exponential distribution.

Apply confidence-based optimization to **other stochastic optimization problems**.

Questions