Global Chance-Constraints: an Application to Stochastic Inventory Control

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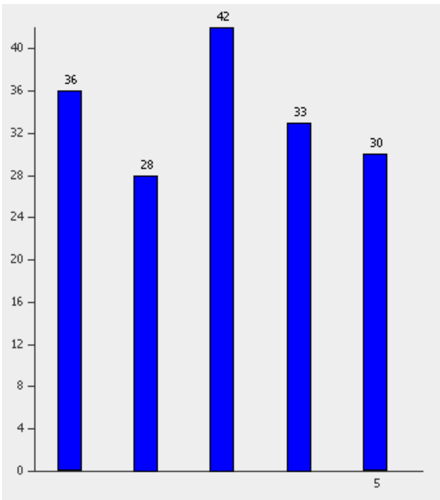
Uncapacitated Deterministic Inventory Control

- Finding an optimal order policy to meet the demand over a given planning horizon minimizing costs
  - Holding cost: 1
  - Fixed ordering cost: 100 (the unit variable cost can be ignored!)
- Can be solved in Polynomial time using Wagner and Whitin (Shortest Path) algorithm [Wagner et al. - 58]



Capacitated Deterministic Inventory Control

- NP-Hard [Florian et al. - 80]: reduction from Subset Sum
- Their proof can be extended to the uncapacitated case under particular cost functions
  - A convex objective function with concave holding cost and fixed ordering cost can be interpreted as a capacity constraint

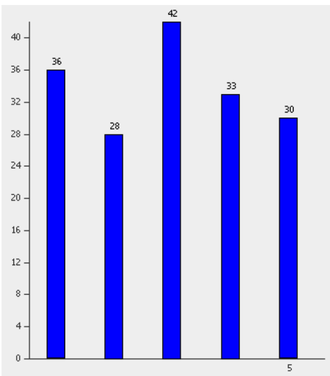
Summary of Complexity Results

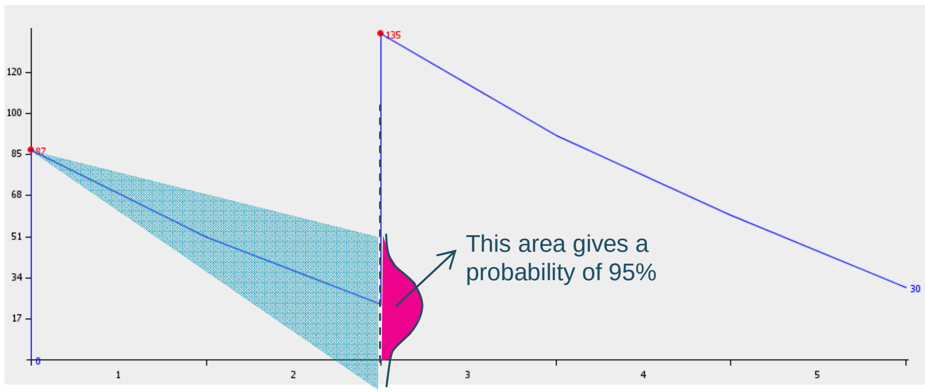
arbitrary cost functions		capacity limits: infinite	capacity limits: equal	capacity limits: arbitrary	convex cost functions
set-up costs	zero	†	†	†	*
	equal	†	†	†	†
	arbitrary	†	†	†	†
concave cost functions		*	*	†	

*solvable in polynomial time
†solvable in pseudopolynomial time, and *NP*-hard even for equal demands and zero storage costs
(x) see §x

Stochastic Inventory Control under Service Level Constraint

- PSPACE – Stochastic Constraint Programming [Walsh - 02]
- Reduction from TQBF $\exists x_1 \forall x_2 \exists x_3 \forall x_4 : (x_1 \vee \neg x_3 \vee x_4) \wedge (\neg x_2 \vee x_3 \vee \neg x_4)$
- Finding an optimal order policy to meet the demand over a given planning horizon minimizing costs and meeting the required service level chance constraint
  - Holding cost: 1
  - Fixed ordering cost: 100 (the unit variable cost cannot be ignored in general!)
  - Service level: 95%
  - Coeff of variation: 0.3





Stochastic Inventory Control under Service Level Constraint

- Chance-constrained model:

minimise $E\{\mathrm{TC}\}$

$$= \int_{d_1}\int_{d_2}\cdots\int_{d_N}\sum_{t=1}^{N}(a\delta_t + hI_t + vX_t\}$$

$$\times\, g_1(d_1)g_2(d_2)\ldots g_N(d_N)\mathrm{d}(d_1)\mathrm{d}(d_2)\ldots\mathrm{d}(d_N) \quad (1)$$

subject to

$$\delta_t = \begin{cases} 1 & \text{if } X_t > 0, \\ 0 & \text{otherwise,} \end{cases} \quad t = 1, \ldots, N, \qquad (2)$$

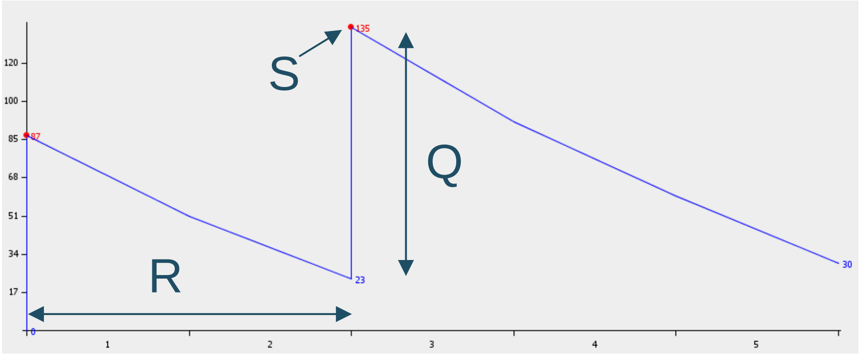
$$I_t = I_0 \sum_{i=1}^{t}(X_i - d_i), \quad t = 1, \ldots, N, \qquad (3)$$

$$\Pr\{I_t \geqslant 0\} \geqslant \alpha, \quad t = 1, \ldots, N, \qquad (4)$$

$$X_t, I_t \geqslant 0, \quad t = 1, \ldots, N,$$

Stochastic Inventory Control under Service Level Constraint

- Different strategies (policies) to model multistage stochastic problems:
  - Observation first
  - Decision first
  - Hybrid strategies
- (R^n, S^n) policy is hybrid: **order-up-to-level** as deterministic decisions



Stochastic Inventory Control under Service Level Constraint

- First complete method for (R^n,S^n): CP approach [Tarim et al. 05]
  - certainty equivalent model (NP-Hardness due to buffer constraints)
  - negative order quantities classified as rare events and therefore ignored

min $\quad E[TC] = \sum_{t=1}^{N} \left(a\delta_t + h\tilde{I}_t \right)$

s.t. $\quad t = 1,\ldots,N$

$$\tilde{I}_t + \tilde{d}_t - \tilde{I}_{t-1} \geq 0$$

$$\tilde{I}_t + \tilde{d}_t - \tilde{I}_{t-1} > 0 \Rightarrow \delta_t = 1$$

$$\tilde{I}_t \geq \Phi[t, \max_{j \in [1..t]} j\delta_j]$$

$$\tilde{I}_t \in Z^+ \bigcup \{0\} \qquad \delta_t \in \{0,1\}$$

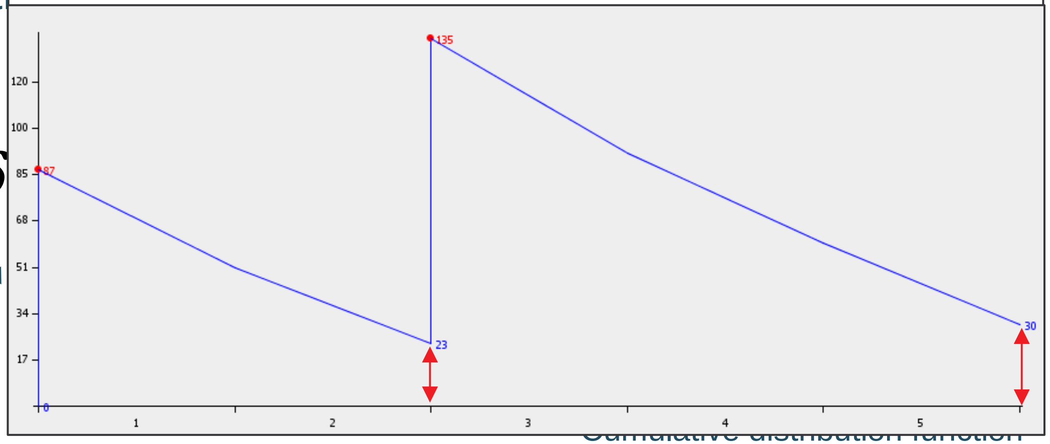
Cumulative distribution function

$$\Phi[i,j] = G^{-1}_{d_j + d_{j+1} + \ldots + d_i}(\alpha) - \sum_{k=j}^{i} \tilde{d}_k$$

Stochastic Inventory Control under Service Level Constraint

- First complete method
  - certainty equivalent r
  - negative order quanti ignored

Identifies the correct **deterministic buffer stock level** needed to satisfy the required service level for a given replenishment cycle



$$\min \quad E[TC] = \sum_{t=1}^{N} \Big(a\delta$$

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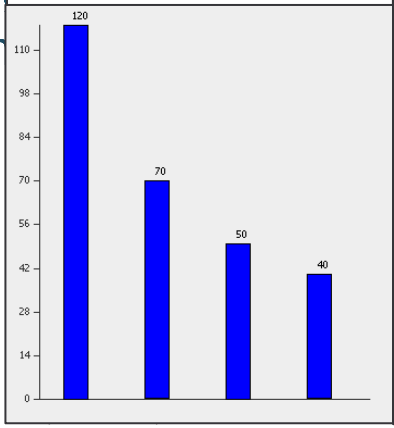
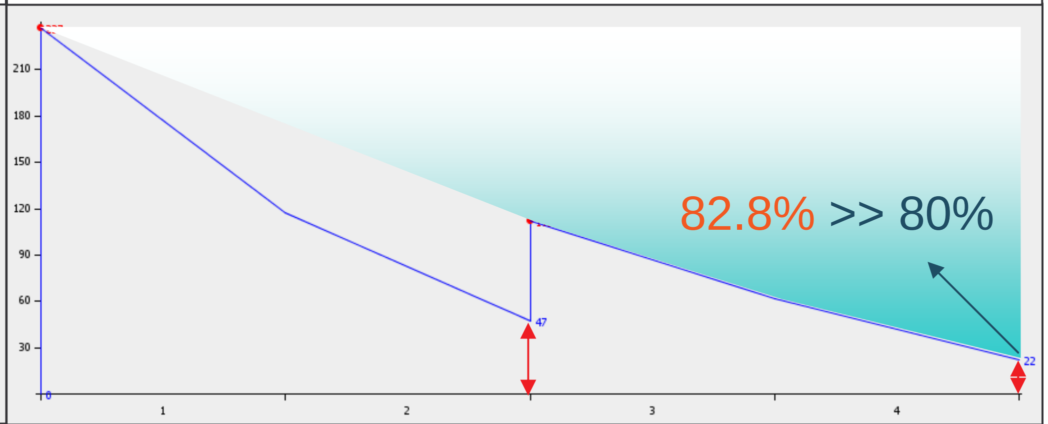
Stochastic Inventory Control under Service Level Constraint

- First c
  - cer
  - ne
    ign

Oc = 150
Hc = 1
Service = 80%
Cv = 0.4

Ignores the effect of *negative order quantities*:
Effect on the *cost*
Effect on the *service level* of following periods





Cumulative distribution function

min

s.t.

$\tilde{I}_t +$

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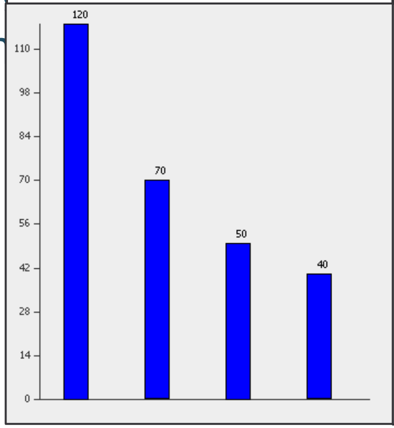

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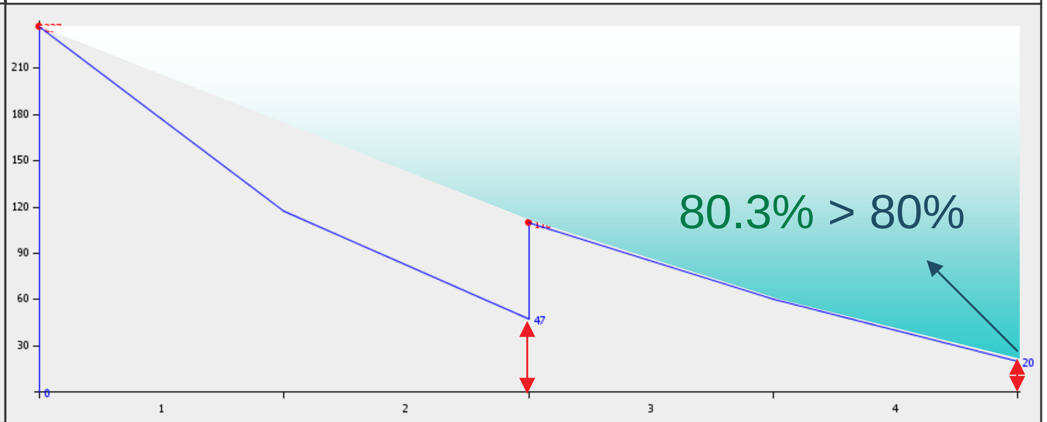

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?

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Stochastic Inventory Control: Global Chance-Constraint

- First c
  - ce
  - ne
    ign

min

s.t.

$\tilde{I}_t +$

$$\tilde{I}_t + \tilde{d}_t - \tilde{I}_{t-1} > 0 \Rightarrow \delta_t = 1$$

$$serviceLevel(\tilde{I}, \delta, d, \alpha)$$

$$\tilde{I}_t \in Z^+ \bigcup \{0\} \qquad \delta_t \in \{0,1\}$$

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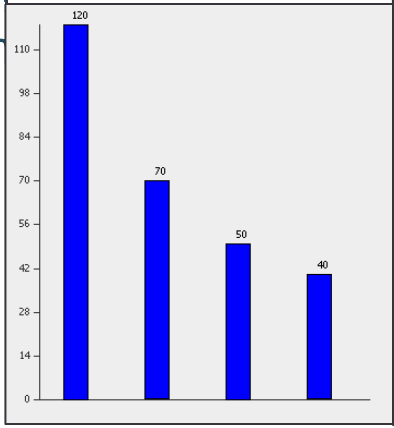
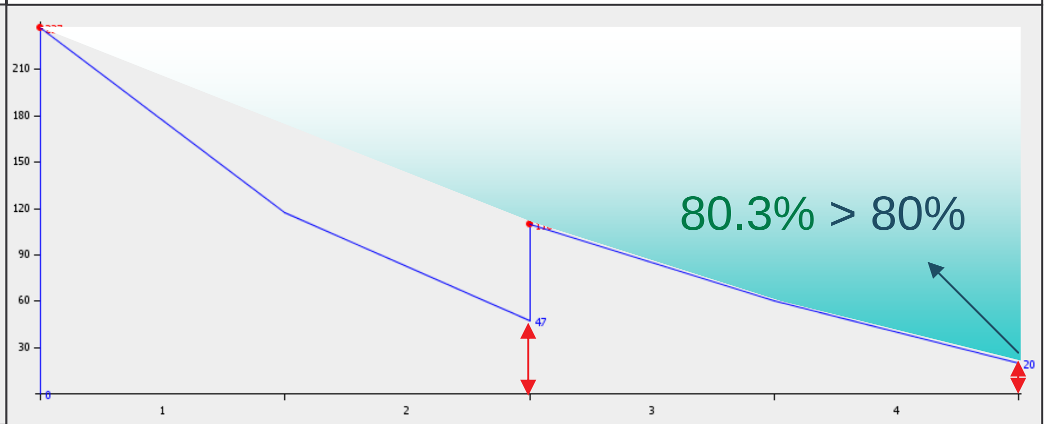




Cumulative distribution function

$$\Phi[i,j] = G^{-1}_{d_j + d_{j+1} + \ldots + d_i}(\alpha) - \sum_{k=j}^{i} \tilde{d}_k$$

Stochastic Inventory Control: Global Chance-Constraint

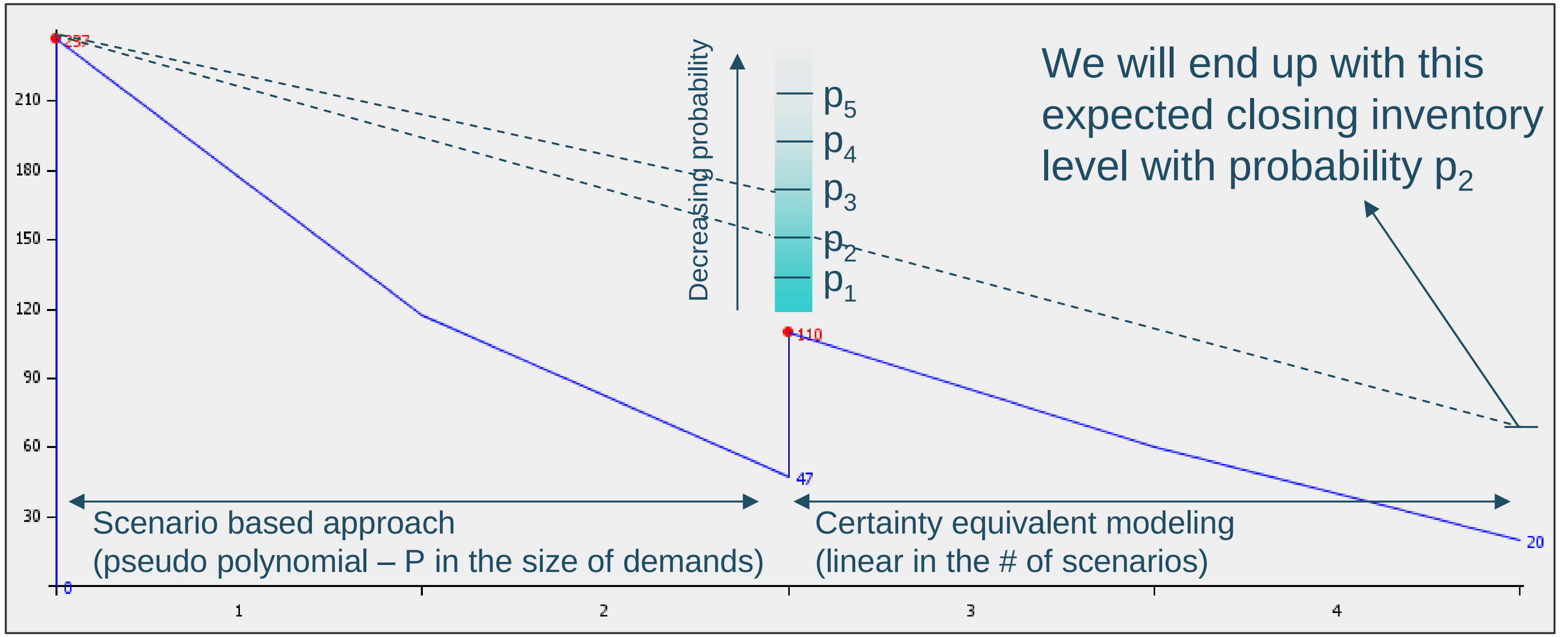
- $serviceLevel(\tilde{I}, \delta, d, \alpha)$ assures that for each replenishment period, at the end of each and every time period, the probability the net inventory will not be negative is at least alpha
  - $\tilde{I}$, δ are arrays of deterministic decision variables
  - d is an array of random variables with p.d.f. $g_i(d_i)$
  - α is the required thresold probability

  - It is therefore semantically equivalent to the original set of chance constraints $\Pr\{I_t \geq 0\} \geq \alpha, t \in \{1, \ldots, N\}$

- The interaction between consecutive replenishment cycles is considered, we are therefore solving the original problem formulation in PSPACE and not a relaxed NP-Hard problem.

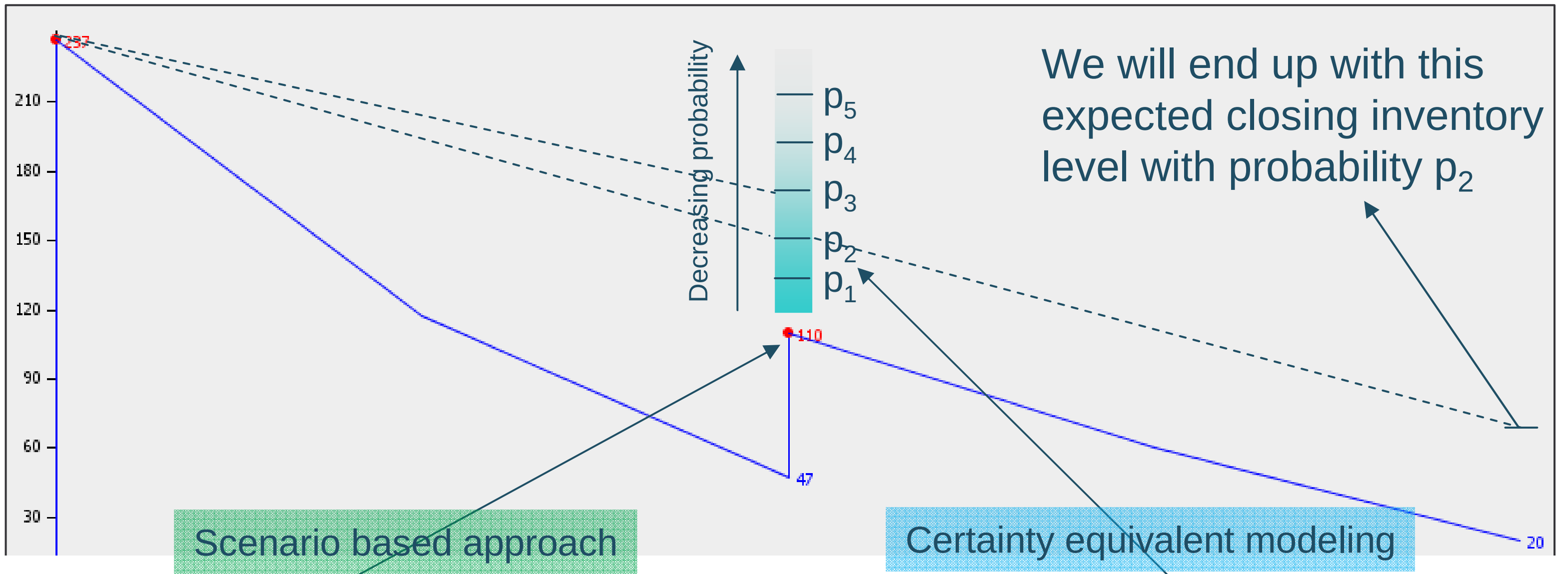
Stochastic Inventory Control: Global Chance-Constraint

- $serviceLevel(\tilde{I}, \delta, d, \alpha)$ computes at each node of the search tree a **convolution integral** that considers joint probabilities of different scenarios which contribute to the overall service level…



Stochastic Inventory Control: Global Chance-Constraint

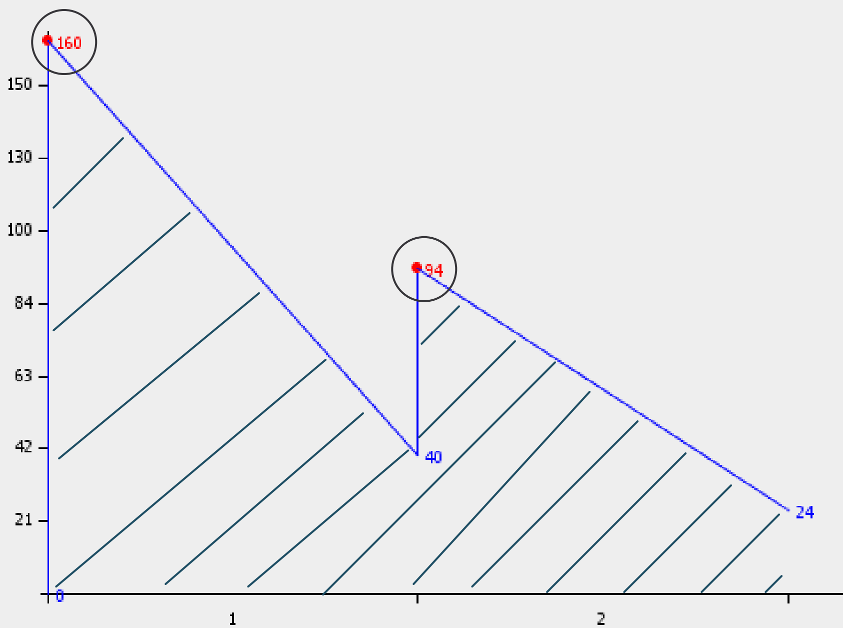
- $serviceLevel(\tilde{I}, \delta, d, \alpha)$ …is therefore able to check if a given configuration for the $\tilde{I}$, δ variables is feasible w.r.t. the required service level constraint and to enforce the condition during search



$$\Pr\{d_1 + d_2 > S_1 - S_2\} \cdot G_{d_3+d_4}(S_2) + \sum_{i=0}^{S_1 - S_2} \left(\Pr\{d_1 + d_2 = i\} \cdot G_{d_3+d_4}(S_1 - i) \right) \geq \alpha$$

Stochastic Inventory Control: Global Chance-Constraint

- $serviceLevel(\tilde{I}, \delta, d, \alpha)$ …is able to detect infeasible partial assignments for $\tilde{I}$, δ variables and to filter domains.



$$\delta_1 = 1, \delta_2 = 1, \delta_3 = -, \delta_4 = -$$
$$I_1 = 40, I_2 = -, I_3 = -, I_4 = -$$

Global Chance-Constraint: Future Extensions

- $serviceLevel(\tilde{I}, \delta, d, \alpha)$
- More effective domain filtering methods and forward checking strategies can be developed
  - ***forward checking based on scenario probabilities***: fails when, observing current assignments, the remaining decision variable assignments are not able to add up to the required thresold: "Stochastic Constraint Programming", [Walsh – 02]
  - ***linear dependency between closing inventory levels for non-replenishment periods****:* not all the former closing inventory levels have to be ground. For each replenishment cycle we require only that at least one $\tilde{I}_t$ variable is ground
  - ***cost based filtering by relaxation****:* the relaxation can be a simple problem (**P**) or again a difficult problem (**NP**-hard), which is easier than our original problem (**PSPACE**). Intuitively the more difficult is the relaxation the tighter will be the bound obtained

Conclusions

- We described an application of Stochastic Constraint Programming in Inventory Control
- We developed global chance-constraints to enforce multiple chance-constraints, when these are mutually dependent in multistage problems
- We showed how to express these complex relations by merging certainty equivalent modeling and scenario based approach during the search
- We suggested strategies to improve the efficiency of this approach in our Inventory Control application