HACETTEPE UNIVERSITY

Replenishment Planning for Stochastic Inventory System with Shortage Cost

Roberto Rossi	UCC, Ireland
S. Armagan Tarim	HU, Turkey
Brahim Hnich	IUE, Turkey
Steven Prestwich	UCC, Ireland

Cork Constraint Computation Centre

ctvr

ucc University College Cork

Inventory Control

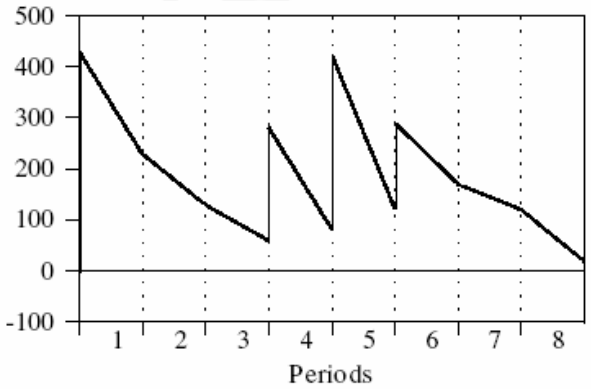
- Computation of optimal replenishment policies under demand uncertainty.

When to order?
How much to order?

?

Demand Uncertainty

Production



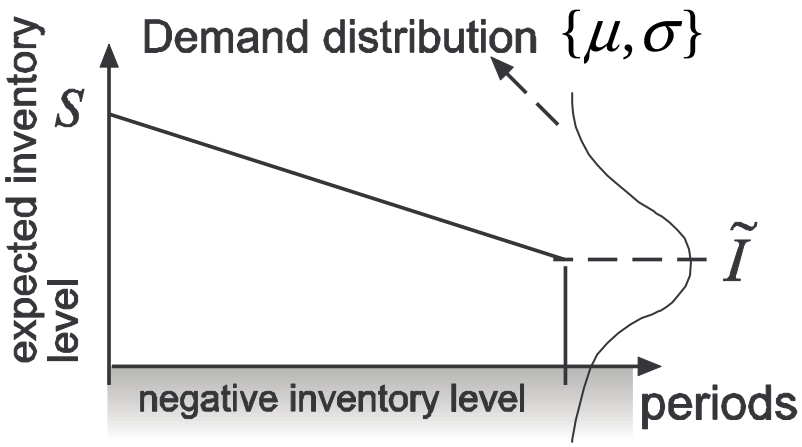
Customers

Newsvendor problem

- We want to determine the optimal quantity of newspaper we should buy in the morning to meet a daily uncertain demand that follows a known distribution (typically normal)
- Two well known approaches: minimize the expected total cost under
  - Service level constraint
  - Shortage cost

Newsvendor problem

- Problem parameters
  - Holding cost h
  - Demand distribution $g(d)$



Service level

service level α

$\Pr\{S \geq d\} \geq \alpha$

$G(S) \geq \alpha$

$S^* = G^{-1}(\alpha)$

$E[TC] = h \int_{t=0}^{S} (S - t) g(t) dt$

Shortage cost

shortage cost s

$z = F_{N(0,1)}^{-1}\left(\frac{s}{s+h}\right)$

$E[TC] = h \int_{t=0}^{S} (S - t) g(t) dt + s \int_{t=S}^{\infty} (t - S) g(t) dt$

$\frac{\partial}{\partial S} E[TC] = h \cdot G(S) - s \cdot (1 - G(S))$ $\quad \frac{\partial^2}{\partial^2 S} E[TC] = h \cdot g(S) + s \cdot g(S) \geq 0$

$G(S^*) = \frac{s}{s+h} (= \alpha)$ $\quad E[TC](S^*) = (h+s) g_{N(0,1)}(z) \cdot \sigma$

Newsvendor problem under shortage cost scheme

- Cost analysis

Let $\frac{S-\mu}{\sigma} = z_\beta$, then for any given S such that $G_{N(0,1)}\left(\frac{S-\mu}{\sigma}\right) = \beta$

we proved that the expected total cost for the single period newsvendor problem can be computed as

$$E[TC](S) = hz_\beta\sigma + (h+s)\sigma[g_{N(0,1)}(z_\beta) - (1-\beta)z_\beta]$$

In the particular case where $\beta = \left(\frac{s}{s+h}\right)$ the ***E*[*TC*]** becomes

$$E[TC](S^*) = (h+s)g_{N(0,1)}(z)\cdot\sigma$$

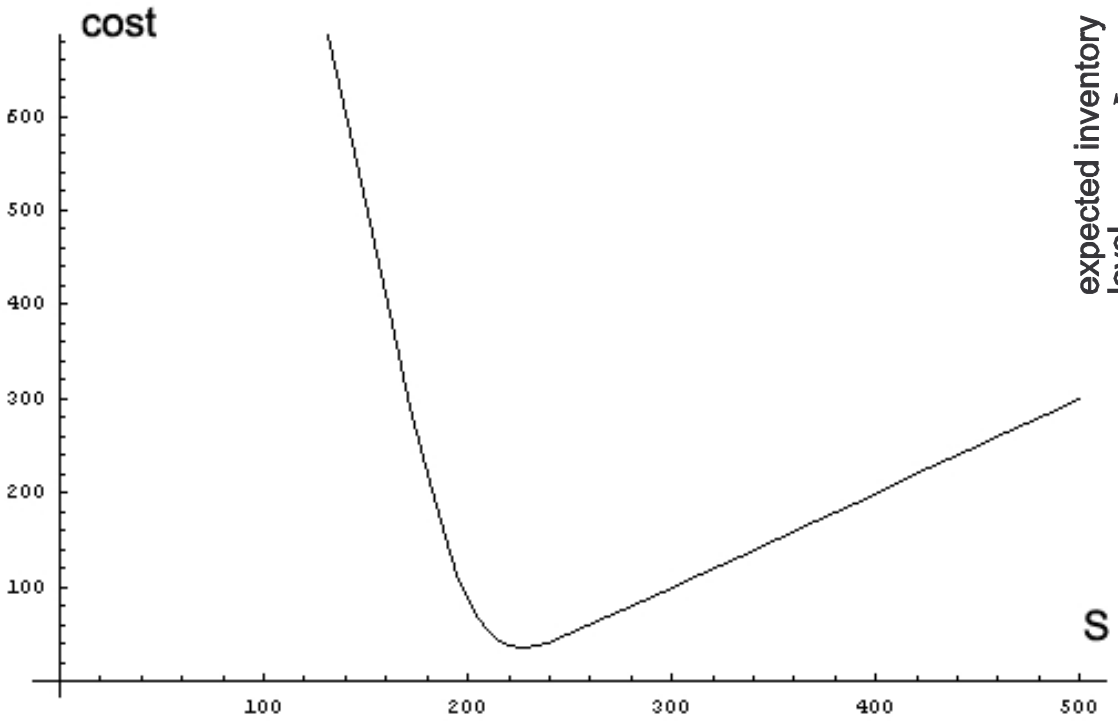
and z, S^* are computed as shown before:

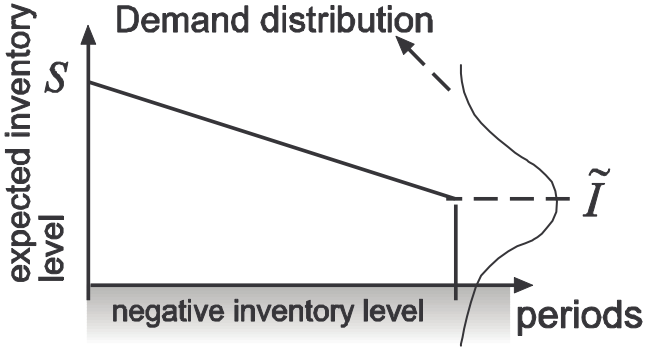
$$z = G^{-1}_{N(0,1)}\left(\frac{s}{s+h}\right) \qquad S^* = \mu + z\sigma = G^{-1}\left(\frac{s}{s+h}\right)$$

Newsvendor problem under shortage cost scheme

- Cost analysis:

$$E[TC](S) = hz_\beta\sigma + (h+s)\sigma[g_{N(0,1)}(z_\beta) - (1-\beta)z_\beta]$$

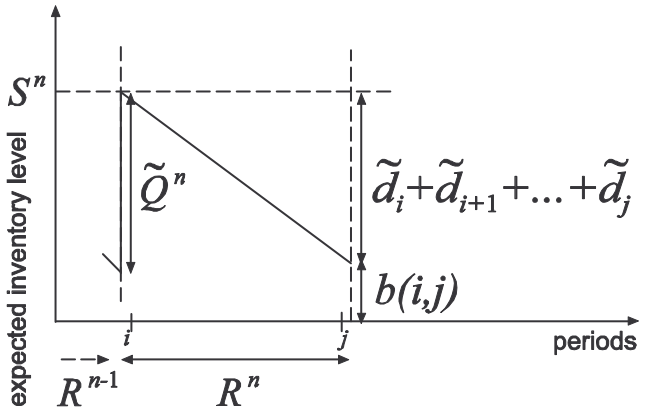




Demand is normally distributed with mean 200 and standard deviation 20. Holding cost is 1, shortage cost is 10.

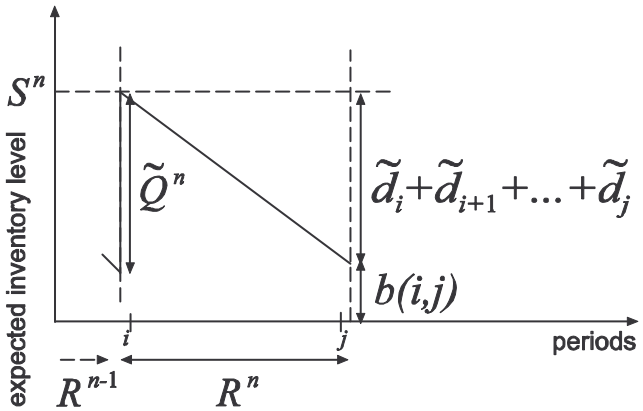
(R^n,S^n) policy under shortage cost scheme

- Replenishment cycle policy (R,S)
  - effective in damping *planning instability,* also known as *control nervousness.*
  - Silver [Sil – 98] points out that this policy is appealing in several cases:
    - Items ordered from the same supplier (joint replenishments)
    - Items with resource sharing
    - Workload prediction
    - *…*
- Dynamic (R,S) [Boo – 88]
  - Considers a non-stationary demand over an N-period planning horizon



(R^n,S^n) policy under shortage cost scheme – assumptions [Tar – 06]

- Dynamic (R,S) [Boo – 88]
  - Considers a non-stationary demand over an N-period planning horizon



- **(1)** Negative orders are not allowed, if the actual stocks exceed the order-up-to-level for a review, this excess stock is carried forward and not returned to the supplier
- **(2)** Such occurrences are regarded as rare events therefore the cost of carrying this excess stock is ignored

(R^n,S^n) policy under shortage cost scheme: stochastic programming model [Tar – 06]

$$\min\ E\{TC\} = \int_{d_1}\int_{d_2}\dots\int_{d_N}\sum_{t=1}^{N}\left(a\delta_t + vX_t + hI_t^+ + sI_t^-\right) g_1(d_1)\dots g_N(d_N)\mathrm{d}(d_1)\dots\mathrm{d}(d_N) \tag{1}$$

subject to

$$X_t > 0 \Rightarrow \delta_t = 1 \tag{2}$$

$$I_t = \sum_{i=1}^{t}(X_i - d_i) \tag{3}$$

$$I_t^+ = \max(0, I_t) \tag{4}$$

$$I_t^- = -\min(0, I_t) \tag{5}$$

$$X_t, I_t^+, I_t^- \in \mathbb{Z}^+ \cup \{0\}, \quad I_t \in \mathbb{Z}, \quad \delta_t \in \{0,1\} \tag{6}$$

for $t = 1 \dots N$, where

d_t : the demand in period t, a normal random variable with PDF $g_t(d_t)$,
a : the fixed ordering cost,
v : the proportional direct item cost,
h : the proportional stock holding cost,
s : the proportional shortage cost,
δ_t : a {0,1} variable that takes the value of 1 if a replenishment occurs in period t and 0 otherwise,
I_t : the inventory level at the end of period t, $-\infty < I_t < +\infty$, $I_0 = 0$
I_t^+ : the excess inventory at the end of period t carried over to the next period, $0 \le I_t^+$,
I_t^- : the shortages at the end of period t, or magnitude of negative inventory $0 \le I_t^-$,
X_t : the replenishment order placed and received in period t, $X_t \ge 0$.

(R^n,S^n) policy under shortage cost scheme: stochastic programming model [Tar – 06]

- The proposed non-stationary (R,S) policy consists of a series of review times (R^n) and order-up-to-levels (S^n).
- We now consider a review schedule which has *m* reviews over an N-period planning horizon with orders arriving at $\{T_1, T_2,\ldots T_m\}$, where $T_j>T_{j-1}$. For convenience we always fix an order in period 1: $T_1=1$.
- In [Tar – 06] the decision variable X_{T_j} is expressed in term of a new variable S_t that may be interpreted as:
  - The opening stock level for period t, if there is no replenishment in this period ($t \neq T_i$)
  - The order-up-to-level for period t if a replenishment is scheduled in such a period ($t = T_i$)

(R^n,S^n) policy under shortage cost scheme: stochastic programming model [Tar – 06]

- According to this transformation, by defining $D_{t_1,t_2} = \sum_{j=t_1}^{t_2} d_j$, the expected total cost in the former model is expressed as

$$\min\ E\{TC\} = \sum_{i=1}^{m} \left(a\delta_{T_i} + \sum_{t=T_i}^{T_{i+1}-1} E\{C_{T_i,t}\} \right) + vI_N + v \int_{D_{1,N}} D_{1,N} \times g(D_{1,N}) \mathrm{d}(D_{1,N}), \tag{7}$$

The term $v \int_{D_{1,N}} D_{1,N} \times g(D_{1,N}) \mathrm{d}(D_{1,N})$ is constant and can therefore be ignored in the optimization model. $E\{C_{T_i,t}\}$ of Eq. (7) is defined as:

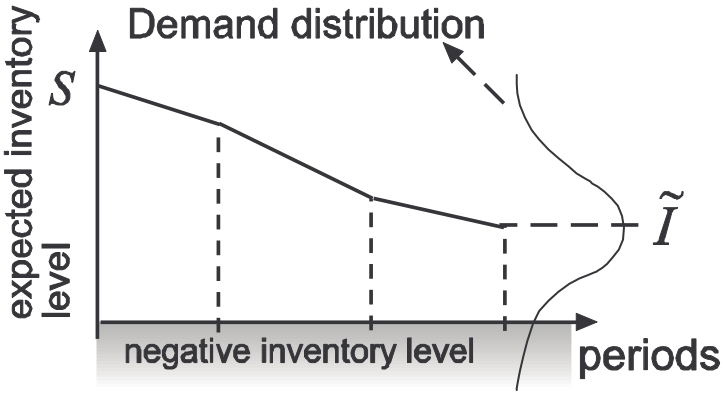
$$\int_{-\infty}^{S_{T_i}} h\,(S_{T_i} - D_{T_i,t})\, g(D_{T_i,t}) \mathrm{d}(D_{T_i,t}) - \int_{S_{T_i}}^{\infty} s\,(S_{T_i} - D_{T_i,t})\, g(D_{T_i,t}) \mathrm{d}(D_{T_i,t}). \tag{8}$$

that is the expected total cost of a single-period newsvendor problem:

$$E\{TC\} = h \int_{-\infty}^{S} (S - D) g(D) \mathrm{d}(D) - s \int_{S}^{\infty} (S - D) g(D) \mathrm{d}(D) \tag{9}$$

Multi-period newsvendor problem under shortage cost scheme

- Expected total cost of a multi-period newsvendor problem



The considerations in the former sections refer to a single-period problem, but they can be easily extended to a replenishment cycle $R(i,j)$ that covers the period span $i,\ldots,j$. The demand in each period is normally distributed with PDF $g_i(d_j),\ldots,g_j(d_j)$. The cost for the multiple periods' replenishment cycle, when ordering costs are neglected, can be expressed as

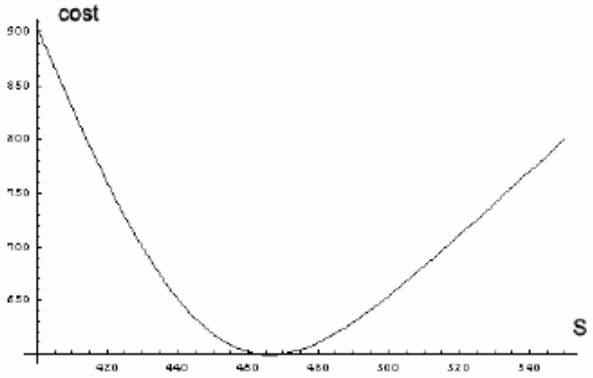
$$E\{TC\} = \sum_{k=i}^{j}\left(h\int_{-\infty}^{S}(S-d_{i,k})g_{i,k}(d_{i,k})\mathrm{d}(d_{i,k}) - s\int_{S}^{\infty}(S-d_{i,k})g_{i,k}(d_{i,k})\mathrm{d}(d_{i,k})\right) \quad (13)$$

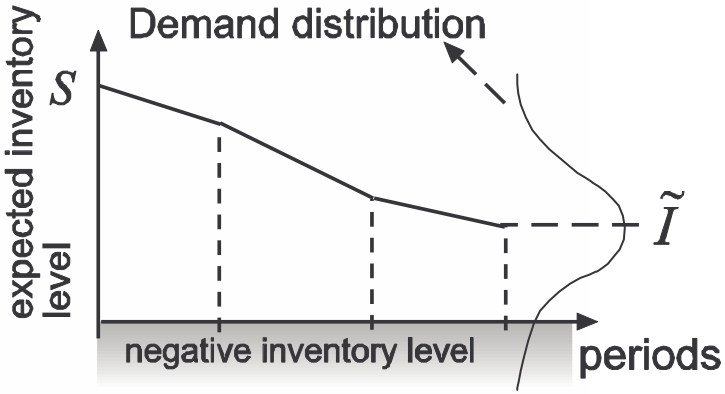
Multi-period newsvendor problem under shortage cost scheme

- By using the closed form expression already presented, the summation becomes:

$$\sum_{k=i}^{j} \left(h z_{\alpha(i,k)} \sigma_{i,k} + (h+s)\sigma_{i,k}[\phi(z_{\alpha(i,k)}) - (1-\alpha(i,k))z_{\alpha(i,k)}] \right) \qquad (15)$$

since the sum of convex functions is a convex function, this expression is convex.





Three periods holding and shortage cost as a function of the opening inventory level S. The demand is normally distributed in each period with mean respectively 150, 100, 200, the coefficient of variation is 0.1. Holding cost is 1, shortage cost is 10.

Multi-period newsvendor problem under shortage cost scheme

- The cost for a replenishment cycle can be expressed as:

$$C(S_i, i, j) = a + \sum_{k=i}^{j} \left(h z_{\alpha(i,k)} \sigma_{i,k} + (h+s)\sigma_{i,k} [\phi(z_{\alpha(i,k)}) - (1 - \alpha(i,k)) z_{\alpha(i,k)}] \right) \tag{17}$$

- **Upper bound** for opening-inventory-levels:
  we optimize the convex cost of $R(1, N)$, this will produce a buffer stock $b(1, N)$. Then for each period $t \in \{1, ..., N\}$,

  $\max(S_t) = \sum_t^N \tilde{d}_t + b(1, N).$

- **Lower bound** for closing-inventory-levels:
  we consider the buffer stock $b(i, j)$: required to optimize the convex cost of each replenishment cycle $R(i, j)$ considered independently on the others. The lower bound is the minimum of these values for $j \in \{1, ..., N\}$ and $i \in \{1, ..., j\}$.

(R^n,S^n) policy under shortage cost scheme: deterministic equivalent model

- A *deterministic equivalent* [Bir – 97] CP formulation is:

$$\min \; E\{TC\} = C \tag{18}$$

subject to

$$\text{objConstraint}\left(C, \tilde{I}_1, \ldots, \tilde{I}_N, \delta_1, \ldots, \delta_N, d_1, \ldots, d_N, a, h, s\right) \tag{19}$$

and for $t = 1 \ldots N$

$$\tilde{I}_t + \tilde{d}_t - \tilde{I}_{t-1} \geq 0 \tag{20}$$

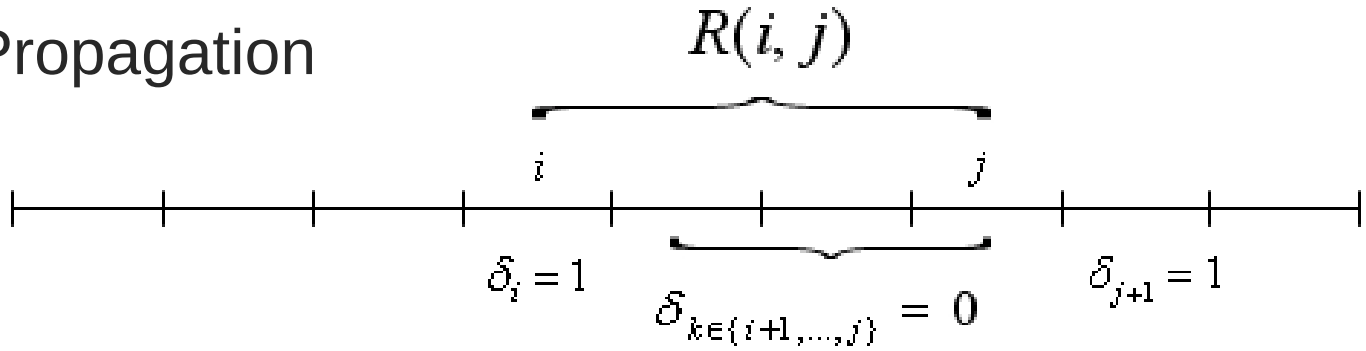
$$\tilde{I}_t + \tilde{d}_t - \tilde{I}_{t-1} > 0 \Rightarrow \delta_t = 1 \tag{21}$$

$$\tilde{I}_t \in \mathbf{Z}, \quad \delta_t \in \{0, 1\} \tag{22}$$

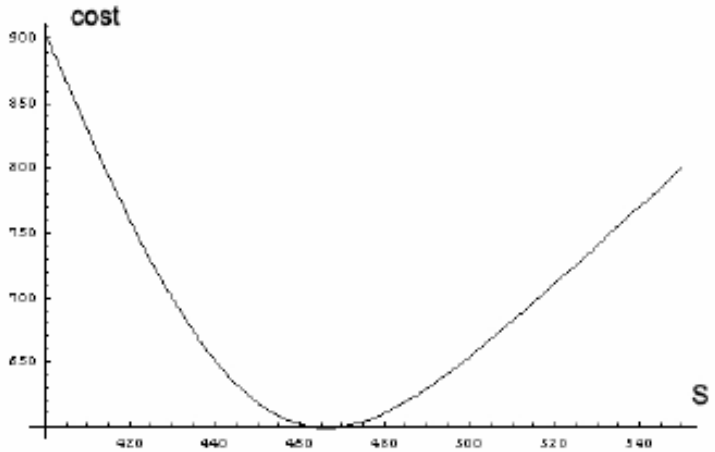
The objective function (18) minimizes the expected total cost over the given planning horizon. $objConstraint(\cdot)$ dynamically computes buffer stocks and it assigns to C the expected total cost related to a given assignment for replenishment decisions, depending on the demand distribution in each period and on the given combination for problem parameters a, h, s.

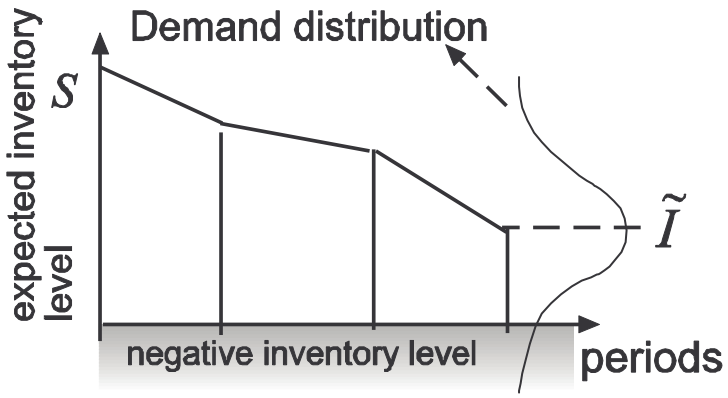
(R^n,S^n) policy under shortage cost scheme: *objConstraint(…)*

- Propagation



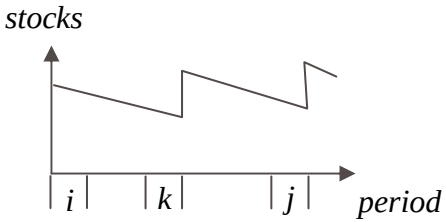
$$C(S_i, i, j) = a + \sum_{k=i}^{j} \left(h z_{\alpha(i,k)} \sigma_{i,k} + (h+s)\sigma_{i,k} [\phi(z_{\alpha(i,k)}) - (1-\alpha(i,k)) z_{\alpha(i,k)}] \right) \quad (17)$$



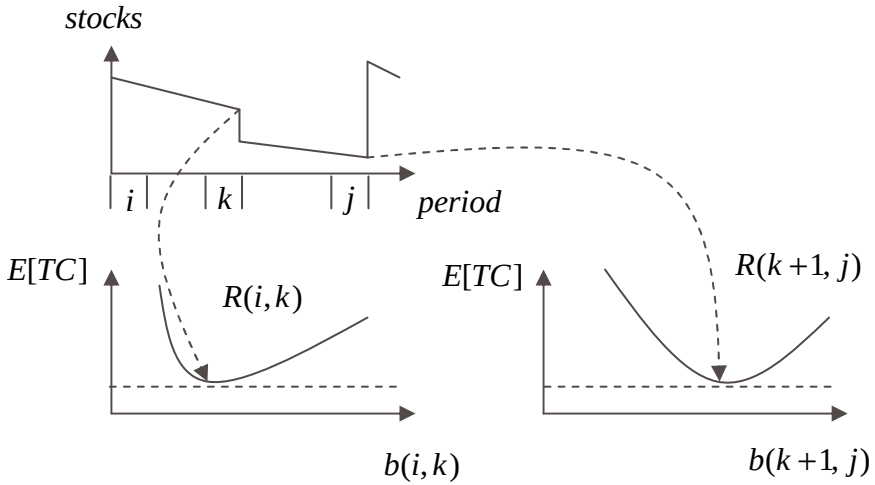


(R^n,S^n) policy under shortage cost scheme: *objConstraint(…)*

- Propagation
  - Inventory conservation constraint met:

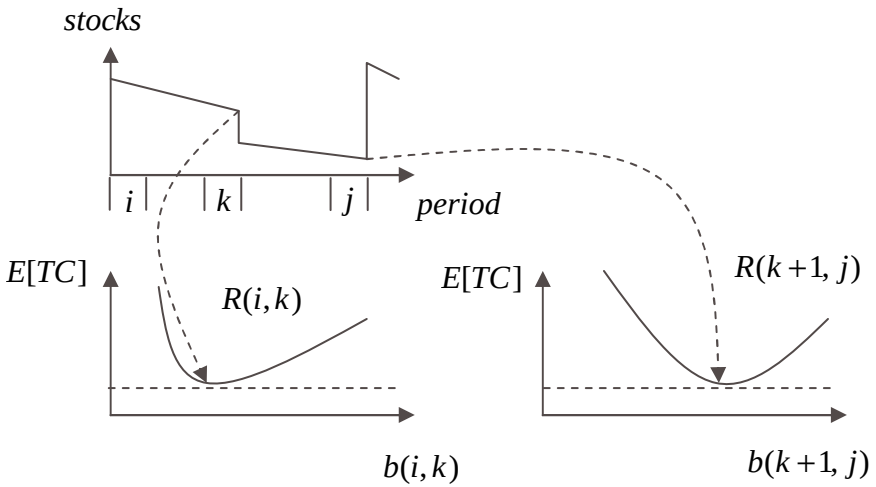


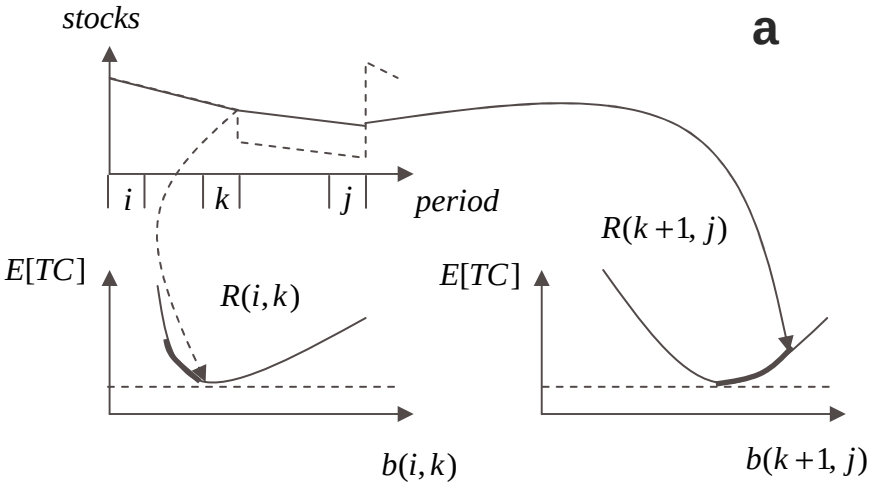
  - Inventory conservation constraint violated:

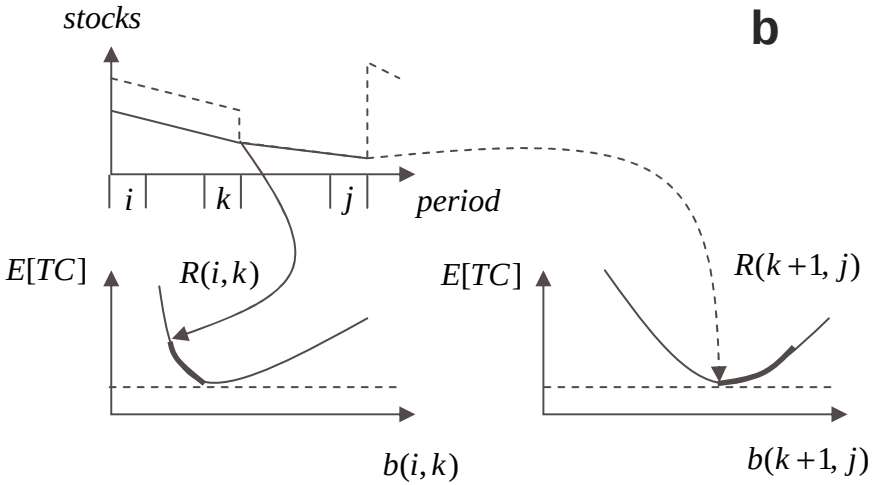


(R^n,S^n) policy under shortage cost scheme: *objConstraint(…)*

- Propagation
  - Inventory conservation constraint violated:

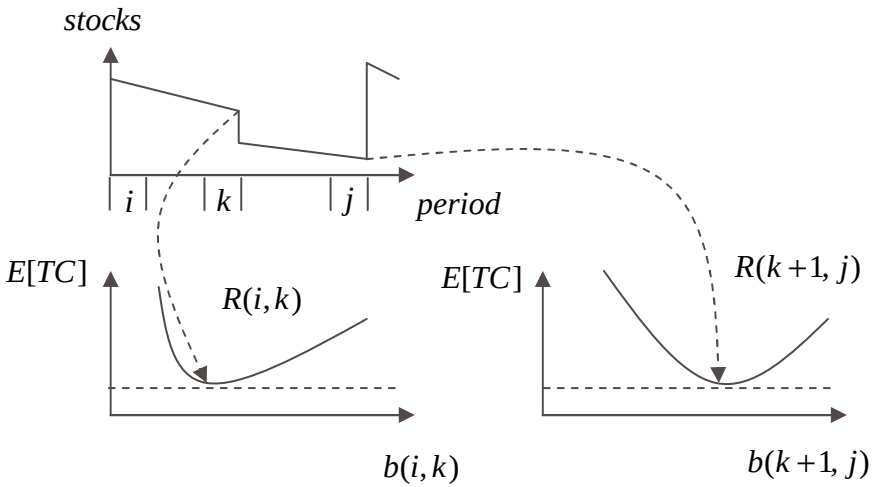


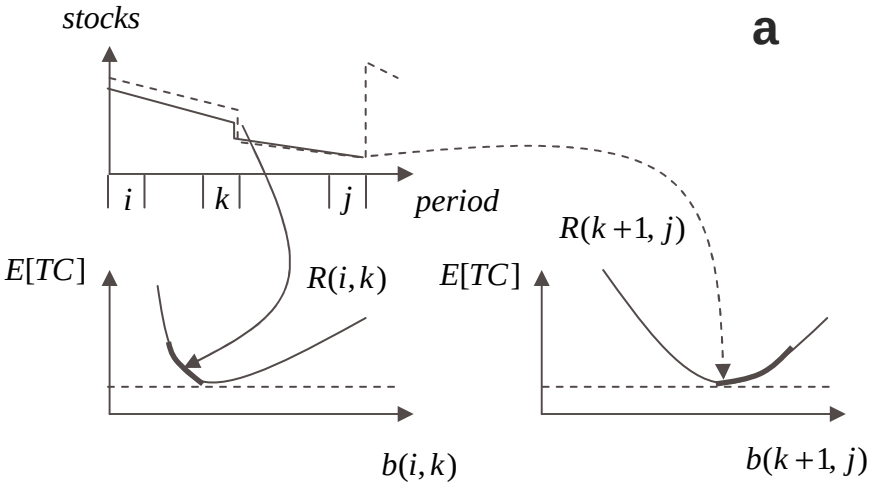


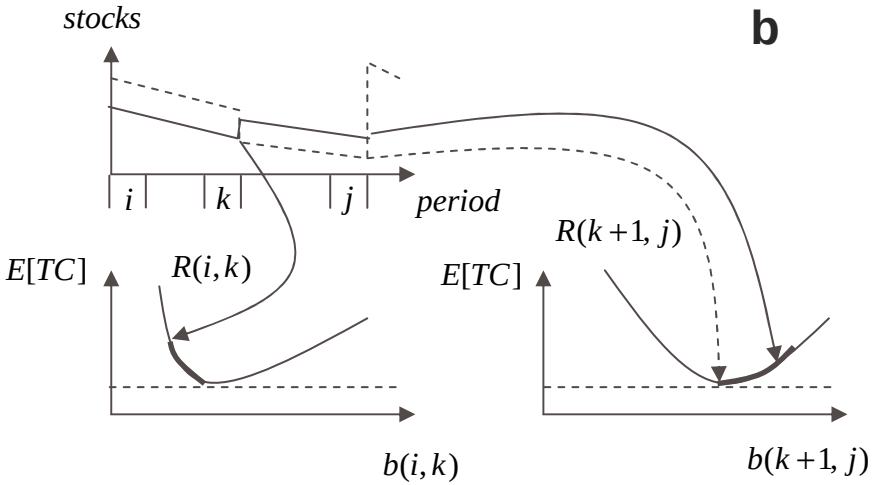


(R^n,S^n) policy under shortage cost scheme: *objConstraint(…)*

- Propagation
  - Inventory conservation constraint violated







(R^n,S^n) policy under shortage cost scheme: Comparison: CP – MIP approach

- We now compare for a set of instances the solution obtained with our CP approach and the one provided by the MIP approach in [Tar – 06]

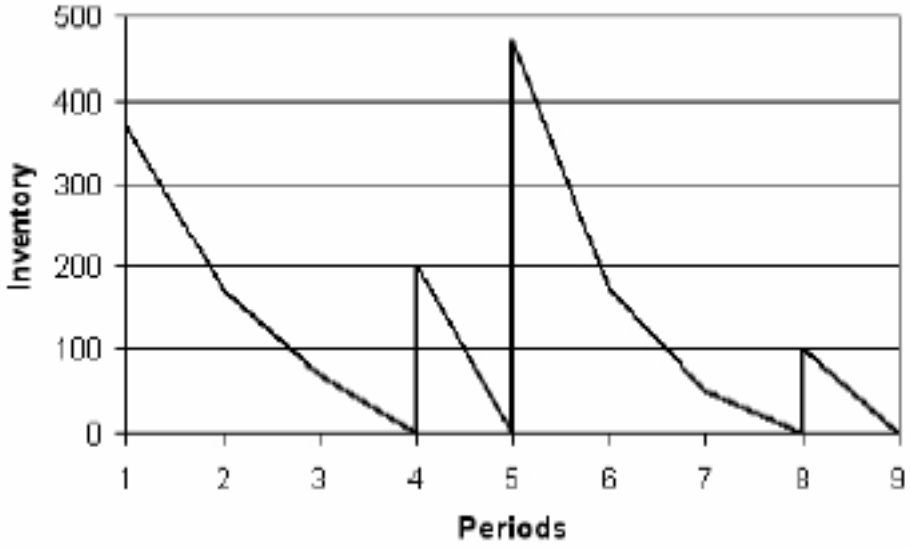
- We consider the following normally distributed demand over an 8-period planning horizon:

Period	1	2	3	4	5	6	7	8
$\bar{d}_t$	200	100	70	200	300	120	50	100

Expected demand values

(R^n,S^n) policy under shortage cost scheme: Comparison: CP – MIP approach

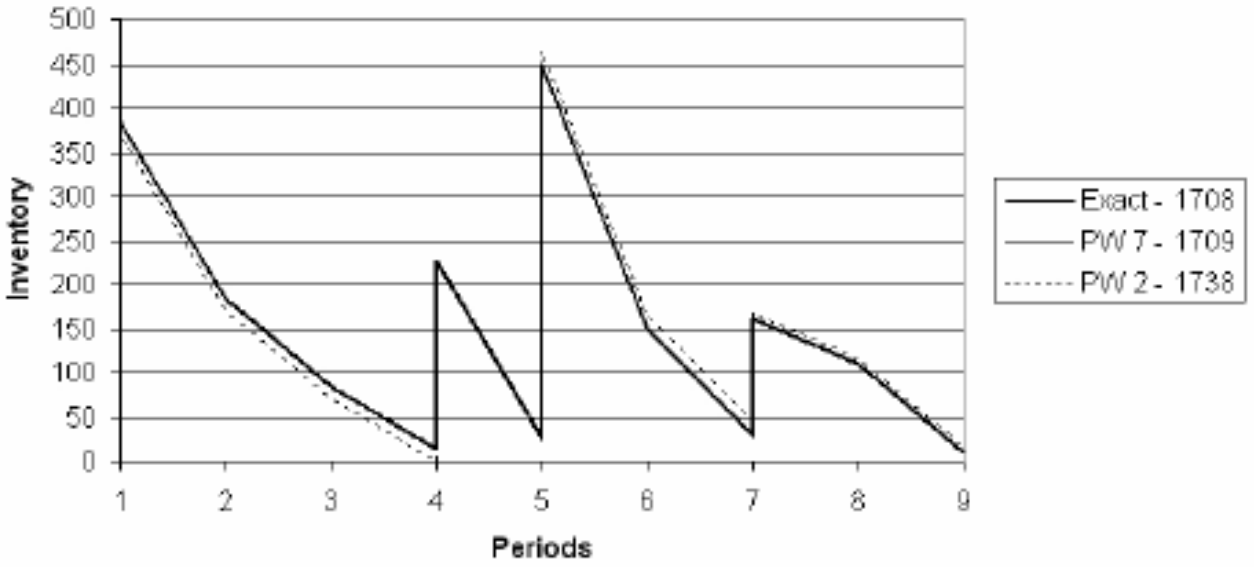
- Deterministic problem [Wag – 58]:



$$h = 1,\ a = 250,\ s = 10,\ v = 0,\ \tau = 0.0$$

(R^n,S^n) policy under shortage cost scheme: Comparison: CP – MIP approach

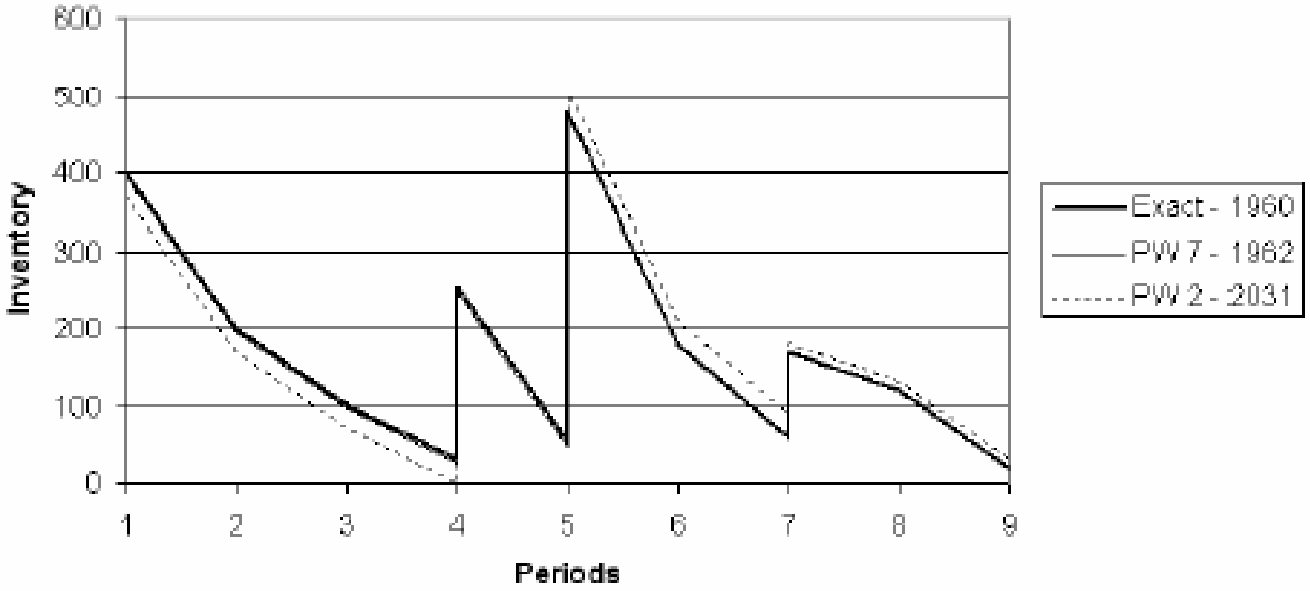
- Stochastic problem. Instance 1 [Tar – 06]:



$$h = 1,\ a = 250,\ s = 10,\ v = 0,\ \tau = 0.1$$

(R^n,S^n) policy under shortage cost scheme: Comparison: CP – MIP approach

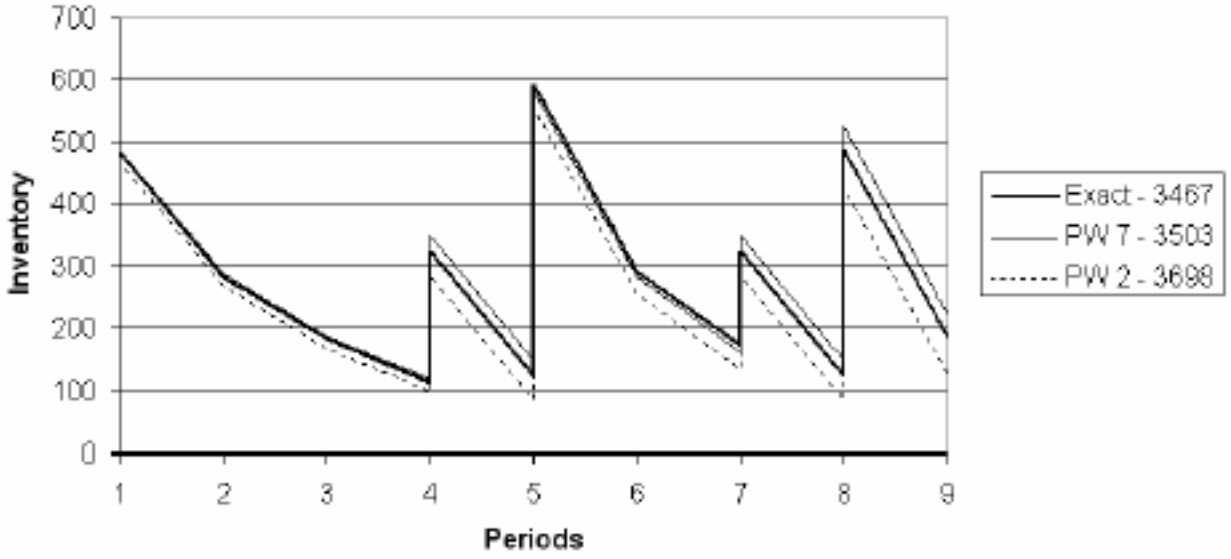
- Stochastic problem. Instance 2 [Tar – 06]:



$$h = 1,\ a = 250,\ s = 10,\ v = 0,\ \tau = 0.2$$

(R^n, S^n) policy under shortage cost scheme: Comparison: CP – MIP approach

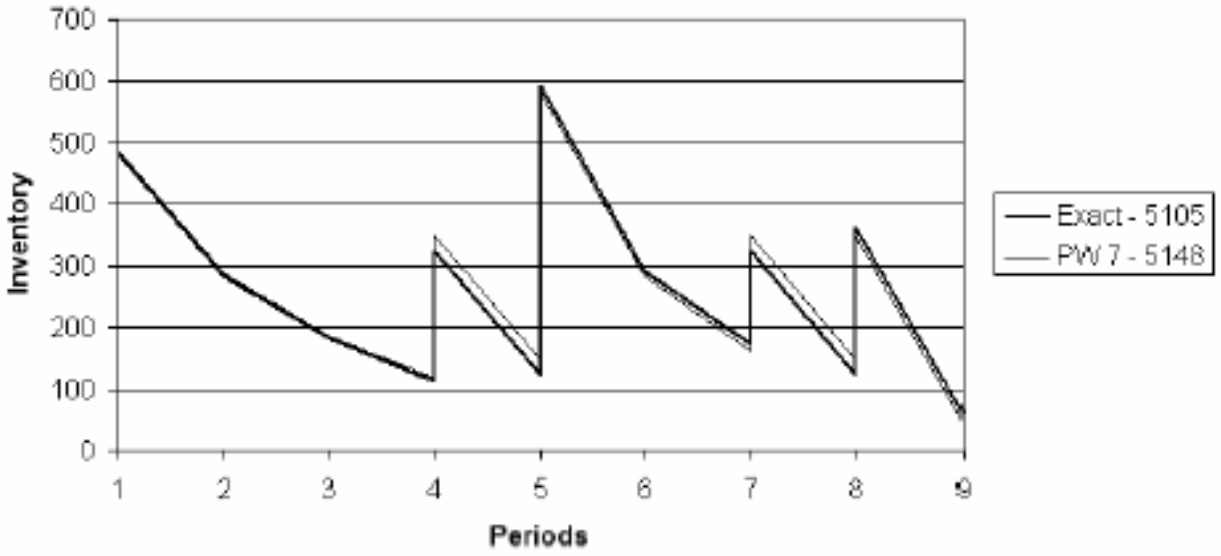
- Stochastic problem. Instance 3 [Tar – 06]:



$$h = 1,\ a = 350,\ s = 50,\ v = 0,\ \tau = 0.3$$

(R^n,S^n) policy under shortage cost scheme: Comparison: CP – MIP approach

- Stochastic problem. Instance 4 [Tar – 06]:



$$h = 1,\ a = 350,\ s = 50,\ v = 15,\ \tau = 0.3$$

(R^n,S^n) policy under shortage cost scheme: CP approach, extensions

- Dedicated cost-based filtering techniques (see [Foc – 99]) can be developed.
- In [Tar – 07] we already presented a similar filtering method under a service level constraint [Tar – 05, Tar – 04].
  - Dynamic programming relaxation [Tar – 96].
- Applying the same technique under a shortage cost scheme requires additional insights, similar to the ones presented in this work, about the convex cost structure of the problem.
- Similar techniques let us solve instances with planning horizons up to 50 periods typically in less than a second for the service level case [Tar – 07].

(R^n,S^n) policy under shortage cost scheme: Conclusions

- We presented a CP approach that finds optimal (R^n,S^n) policies under nonstationary demands.
- Using our approach it is now possible to evaluate the quality of a previously published MIP-based approximation method, which is typically faster than the pure CP approach.
- Using a set of problem instances we showed that a piecewise approximation with seven segments usually provides good quality solutions, while using only two segments can yield solutions that differ significantly from the optimal.
- In future work we will aim to develop domain reduction techniques and cost-based filtering methods to enhance the performance of our exact CP approach.

(R^n,S^n) policy under shortage cost scheme: References

- **[Bir – 97]** J. R. Birge, F. Louveaux. *Introduction to Stochastic Programming*. Springer Verlag, New York , 1997.
- **[Boo – 88]** J. H. Bookbinder, J. Y. Tan. Strategies for the Probabilistic Lot-Sizing Problem With Service-Level Constraints. Management Science 34:1096–1108, 1988.
- **[Foc – 99]** F. Focacci, A. Lodi, M. Milano. *Cost-Based Domain Filtering*. Fifth International Conference on the Principles and Practice of Constraint Programming, Lecture Notes in Computer Science 1713, Springer Verlag, 1999, pp. 189–203.
- **[Sil – 98]** E. A. Silver, D. F. Pyke, R. Peterson. *Inventory Management and Production Planning and Scheduling*. John Wiley and Sons, New York, 1998.
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