Estimating the value of e by simulation

The probability density function of the continuous uniform distribution is:

$$f(x) = \begin{cases} \frac{1}{b-a} & a \leq x \leq b \\ 0 & x < a \text{ or } x > b. \end{cases}$$

The Irwin-Hall distribution is the continuous probability distribution for the sum of n independent and identically distributed continuous uniform random variables U_k with parameters $a = 0$ and $b = 1$. Let $X = \sum_{k=1}^{n} U_k$; the cumulative distribution function of X is given by

$$F_X(x; n) = \frac{1}{n!} \sum_{k=0}^{\lfloor x \rfloor} (-1)^k \binom{n}{k} (x-k)^n.$$

Let $F_X^1(n) \triangleq F_X(1; n)$, then

$$\begin{aligned} F_X^1(n) &= \frac{1}{n!} \sum_{k=0}^{1} (-1)^k \binom{n}{k} (1-k)^n \\ &= \frac{1}{n!} \left((-1)^0 \binom{n}{0} (1-0)^n + (-1)^1 \binom{n}{1} (1-1)^n \right) = \frac{1}{n!}. \end{aligned}$$

Lemma 1 ([1])**.** *Let* $V = \min\{n | U_1 + U_2 + \ldots + U_n > 1\}$*, then* $E[V] = e$.

Proof. Let $\Pr(A)$ be the probability of A and recall $\Pr(A \cap B) = \Pr(A|B)\Pr(B)$.

$$\begin{aligned} &\Pr(U_1 + U_2 + \ldots + U_n > 1 \cap U_1 + U_2 + \ldots + U_{n-1} \leq 1) \\ &= \Pr(U_1 + U_2 + \ldots + U_{n-1} \leq 1) - \Pr(U_1 + U_2 + \ldots + U_n \leq 1) \\ &= F_X^1(n-1) - F_X^1(n) = \frac{n-1}{n!}. \end{aligned}$$

Then

$$\mathrm{E}[V] = \sum_{n=2}^{\infty} n(F_X^1(n-1) - F_X^1(n)) = \sum_{n=2}^{\infty} n \frac{n-1}{n!} = \sum_{n=0}^{\infty} \frac{1}{n!} = e.$$

□

References

[1] K. G. Russell. Estimating the value of e by simulation. *The American Statistician*, 45(1):66, February 1991.