

A cash constrained single item stochastic lot-sizing problem

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Abstract: We investigate a dynamic inventory management problem where a cash-constrained small retailer periodically purchases an item from a supplier and sells it to the customers with non-stationary demands. At each period, the retailer's available cash restricts the maximum inventory level that it can replenish. There exists a fixed ordering cost for the retailer when ordering and this results in a stochastic lot-sizing problem. We build a stochastic dynamic programming model for this problem and find some characteristics of the optimal ordering pattern.

Keywords: inventory control; stochastic lot-sizing; non-stationary demand; cash constraint

1. INTRODUCTION

Cash flow management is important to the survival and growth of many businesses, especially for small-to-medium-sized enterprises (SMEs), including startup companies and small retailers. Cash shortage may disrupt a firm's smooth operation and can even lead to insolvency (John, 2014). A report compiled by the research firm CB Insights found that 29% of startups failed because of cash crisis (CBInsights, 2018). There is one kind of retailers called nanostores, which are family owned in many developing countries (Boulaksil and van Wijk, 2018). Those retailers can only purchase an amount of products according to its available cash. Many small retailers doing business in Chinese E-commerce platforms like JD.com and Taobao.com are similar to the nanostores. Most of these retailers are very small and operated by several people or even individuals. For the nanostores and small online retailers, it is more difficult to obtain external financing compared with larger business entities. Therefore, it is necessary for them to consider cash constraints during business operations.

Cash constraints in the inventory management problems have been investigated by many pioneering works. Here we list several of them. Buzacott and Zhang (2004) pointed out the importance of jointly considering operation and financial decisions by analyzing a cash constrained single-period newsvendor model with financing behaviors. They showed the optimal policy is presented in terms of two thresholds and also propose a solving algorithm. Dada and Hu (2008) build a Stackelberg model between a bank, manufacturer and cash-constrained retailer. Raghavan and Mishra (2011) considered a two-level supply chain with a retailer and manufacturer both facing cash constraints. Kouvelis and Zhao (2012) gave a detailed discussion about optimal trade credit contracts in a game model. Moussawi-Haidar and Jaber (2013) also developed a model for a cash constrained retailer under delay payments. Tunca and Zhu

(2017) discussed the role and efficiency of buyer intermediation in supplier financing through a game-theoretical model.

Regarding multi-period inventory problems, Chao et al. (2008) investigated a multi-period self-financing newsvendor problem. They proved the optimal ordering pattern is a base stock policy and presented a simple algorithm to solve the problem for stationary demands. Gong et al. (2014) extended this by considering short-term financing in the model. Katehakis et al. (2016) analyzed non-stationary demand processes and time-varying interests. Boulaksil and van Wijk (2018) formulated a cash-constrained stochastic inventory model with consumer loans and supplier credits for nanostores, and obtain some managerial insights by simulating the numerical cases.

Our work also addresses the multi-period stochastic inventory problem: a small retailer purchases a product from its supplier periodically and sells it to customers. The difference from previous works is that we consider fixed ordering costs for the retailer. Fixed ordering cost does exist for some retailers. For example, an online retailer selling clothes on Taobao.com typically purchases from clothes distributors monthly. The transportation cost and some other expenses of each procurement can be viewed as a fixed ordering cost because they are not related with the ordering quantity. Motivated by this background, we formulate a single-item stochastic lot-sizing model for the problem.

Cash constraint is a kind of capacity constraint which restricts the maximum inventory replenishment level for a retailer. Without capacity constraint, Scarf (1960) proved the optimal ordering policy for the general single item stochastic lot sizing problem is (s, S) , in which s denotes the reorder point and S is the order-up-to-level. The optimality is proved through a property called K -convexity. Regarding the capacitated stochastic lot-sizing problem,

optimal ordering patterns has not been thoroughly characterized. A key finding by Chen and Lambrecht (1996) proved that, with stationary demand and fixed capacity, the optimal policy shows a pattern of $\mathbf{X} - \mathbf{Y}$ band structure: when initial inventory level is below \mathbf{X} , it is optimal to order at full capacity; when initial inventory level is above \mathbf{Y} , do not order. Shaoxiang (2004) further proved this pattern by giving the definition of (C, K) -convex. Chao and Zipkin (2008) gave an optimal policy for a special supply chain contract: fixed ordering cost triggered when ordering quantity is above the contract volume. Özener et al. (2014) relaxed the problem with linear holding and penalty cost, Poisson demand, and proposed a heuristic (s, Δ) policy. They modeling the problem as a discrete-time Markov chain for stationary and infinite horizon. There are also some other works like Chan and Song (2003), Shi et al. (2014), Yang et al. (2014), etc.

For the general (s, S) policy, the computation for the values of s, S of each period has been considered prohibitively expensive for years. A number of exiting works attempted to solve this problem under stationary demands. See Federgruen and Zipkin (1984), Zheng and Federgruen (1991), Feng and Xiao (2000), etc. For non-stationary demands, some recent progress are made by Bollapragada and Morton (1999), Xiang et al. (2018). Nevertheless, those works are for the uncapacitated situation — there are no bounds for the ordering quantity.

To the best of our knowledge, discussion about the optimal policy and computation for the non-stationary stochastic lot sizing problem under cash constraints has not been investigated. This motivates our study to investigate this topic. Our contributions to the literature are as follows.

- Cash constraints are considered in the stochastic multi-period inventory problem, where there are fixed ordering cost for the retailer and maximum ordering quantity is bounded by available cash. This model is helpful for self-financing retailers like nanostores or small online retailers to make inventory replenishment decisions.
- We discuss the characteristics of the optimal ordering policy for the problem with non-stationary demands and finite horizon, and find some characteristics about the optimal ordering pattern.

2. PROBLEM DESCRIPTION

In our problem, a cash constrained retailer orders an item from its suppliers, and sell them to customers. A fixed ordering cost K is charged to the retailer when it orders from the suppliers. Customer's demand is stochastic, non-stationary, independently and identically distributed from period to period. For each period n , its demand is represented by ξ_n , which is a non-negative random variable, the probability density function for ξ_n is $\phi_n(\xi)$ and cumulative distribution function is $\Phi_n(\xi)$. Since the retailer is small in scale, its customers do not wait for the back-ordered items and unmet demand is lost. Excess stock is transferred to the next period as inventory and the sell back of excess stock is not allowed. We assume the maximum possible demand quantity is D_{max} , and minimum possible demand quantity is D_{min} .

Initial cash balance of the retailer is R_0 ; order delivery lead time is zero; selling price of the item is p and the retailer receives payments only when its items are delivered to the customers. A fixed cost K is charged when placing orders by the retailer, regardless of the order quantity. $\delta(Q)$ is a unit step function to determine whether the retailer makes order at period n : $\delta(Q) = 1$ when $Q > 0$, $\delta(Q) = 0$ when $Q = 0$; a variable cost c is charged on every order unit. End-of-period inventory level for period n is x_n , and we set x_n^+ to represent $\max\{x_n, 0\}$. A variable inventory holding cost h is charged on every item unit carried from one period to the next. At each period n , its initial cash is R_{n-1} , ordering quantity is Q_n , and the total ordering cost is $K\delta(Q_n) + cQ_n$, which includes fixed ordering cost and variable ordering costs. Because of cash constraint and immediate payment requirement by the suppliers, when the retailer makes orders, its available cash should be greater than the total ordering cost, which can be represented by the inequality constraint below.

$$K\delta(Q_n) + cQ_n \leq R_{n-1} \quad (1)$$

The difference of our problem with prior inventory literature considering cash constraints like Chao et al. (2008), Gong et al. (2014) is that we consider fixed ordering cost K for the retailer. Another difference is that we do not take account of the deposit interests of each period's cash balance, because the deposit interests from the bank is generally very small compared with the retailer's transaction volumes and usually does not exert an effect on its operational decisions.

The actual sales quantity in period n is $\min\{\xi_n, Q_n + x_{n-1}^+\}$, where ξ_n is the demand quantity in period n and $Q_n + x_{n-1}^+$ is the total available stock in period n . End-of-period cash R_n for period n is defined as its real initial cash balance R_{n-1} , plus payments by customers for the realized demand in this period $p \min\{\xi_n, Q_n + x_{n-1}^+\}$, minus this period's fixed ordering cost, variable ordering cost and holding cost $cQ_n + K\delta(Q_n) + hx_n^+$. Full expression of R_n is given by Eq. (2). The inflows and outflows of cash from period $n - 1$ to period $n + 1$ is detailed shown by Figure 1.

$$R_n = R_{n-1} + p \min\{\xi_n, Q_n + x_{n-1}^+\} - (cQ_n + K\delta(Q_n) + hx_n^+) \quad (2)$$

Any inventory left at the end of the planning horizon has a unit salvage value γ per unit. We also assume $0 < \gamma \leq c < p$. Our aim is to find a replenishment plan that maximizes the expected terminal cash increment, i.e. $E(R_n) + \gamma x_n - R_0$.

3. STOCHASTIC DYNAMIC PROGRAMMING MODELING

In this section, we formulate a stochastic dynamic programming (SDP) model for our problem.

States. The system state at the beginning of period n is represented by initial inventory x_{n-1} and initial cash quantity R_{n-1} .

Actions. The action at period n is the ordering quantity Q_n , given initial inventory x_{n-1} and initial cash quantity R_{n-1} . Because of cash constraint, the upper bound for Q_n can be represented by Con (3).

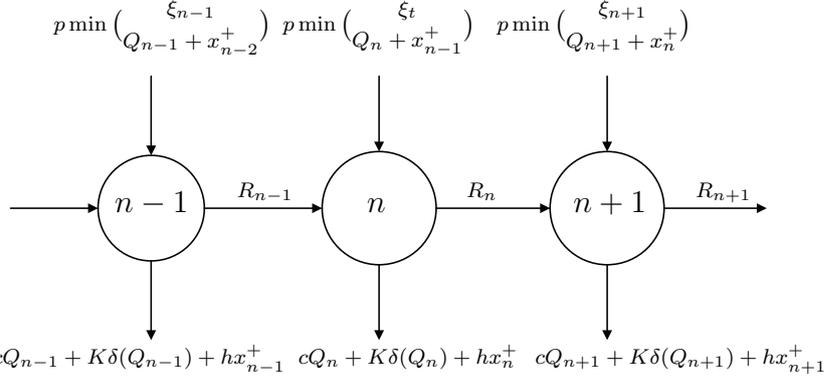


Fig. 1. Inflows and outflows of cash at periods $n - 1, n, n + 1$

$$Q_n \leq \bar{Q}_n = \max \left\{ 0, \frac{R_{n-1} - K}{c} \right\} \quad (3)$$

It is convenient to select the order-up-to level y_n (immediate inventory level after ordering) to replace Q_n as decision variable. $y_n = x_{n-1} + Q_n$. The bounds for y_n is represented by

$$x_{n-1} \leq y_n \leq \bar{y}_n = x_{n-1} + \max \left\{ 0, \frac{R_{n-1} - K}{c} \right\} \quad (4)$$

State transition function. The inventory and cash at the end of period n is determined by initial inventory x_{n-1} , initial cash R_{n-1} , demand ξ_n and action y_n . Let ΔR_n be the cash increment in period n .

$$\Delta R_n = p \min \{ \xi_n, y_n \} - [K\delta(y_n - x_{n-1}) + c(y_n - x_{n-1}) + h(y_n - \xi_n)^+] \quad (5)$$

The state transition of the system for R_n and x_n is described by the following equations.

$$x_n = \max \{ y_n - \xi_n, 0 \} = (y_n - \xi_n)^+ \quad (6)$$

$$R_n = R_{n-1} + \Delta R_n = R_{n-1} + p \min \{ \xi_n, y_n \} - [K\delta(y_n - x_{n-1}) + c(y_n - x_{n-1}) + h(y_n - \xi_n)^+] \quad (7)$$

Immediate profit. The immediate profit for period n is the expected cash increment during this period. Given R_{n-1} , x_{n-1} and y_n , $E(\Delta R_n)$ can be expressed as

$$E(\Delta R_n) = \int \left\{ p \min \{ \xi_n, y_n \} - [K\delta(y_n - x_{n-1}) + c(y_n - x_{n-1}) + h(y_n - \xi_n)^+] \right\} \phi(\xi_n) d\xi_n \quad (8)$$

Optimality Equation. Define $F_n(R_{n-1}, x_{n-1})$ as the maximum expected cash increment during periods $n, n + 1, \dots, N$, when the initial inventory and cash of period n are x_{n-1} and R_{n-1} , respectively. The optimality equation for the problem is expressed as:

$$F_n(x_{n-1}, R_{n-1}) = \max_{x_{n-1} \leq y_n \leq \bar{y}_n} \{ E(\Delta R_n) + \int F_{n+1}((y_n - \xi_n)^+, R_{n-1} + \Delta R_n) \phi(\xi_n) d\xi_n \} \quad (9)$$

$n = 1, 2, \dots, N,$

The boundary equation is:

$$F_{N+1}(x_N, R_N) = \gamma x_N. \quad (10)$$

For convenience of analysis, we define $H_n(y)$ and $L_n(y)$ as follows.

$$H_n(y) = p \min \{ \xi_n, y \} - h(y - \xi)^+ - cy, \quad (11)$$

$$L_n(y) = \int_{D_{min}}^{D_{max}} H(y) \phi(\xi_n) d\xi = \int [p \min \{ \xi_n, y \} - h(y - \xi)^+ - cy] \phi(\xi_n) d\xi, \quad (12)$$

$L_n(y)$ is the one period expected revenue minus holding cost and variable ordering cost. It can be easily proved that $L_n(y)$ is a concave function.

Lemma 1. $L_n(y)$ is a concave function and reaches maximum at

$$x_L = \Phi_n^{-1} \left(\frac{p - c}{p + h} \right) \quad (13)$$

Proof.

$$\frac{d(L_n(y))}{dy} = p - c - (p + h) \int_{D_{min}}^y \phi_n(\xi) dt,$$

$$\frac{d^2(L_n(y))}{dy^2} = -(p + h) \phi_n(y) < 0,$$

Therefore, $L_n(y)$ is a concave function. Letting its first-order derivative be zero, we can get its maximum point at x_L . \square

Usually the specific period n is immaterial, so we suppress n in some expressions, and use it when needing to specify some period. ΔR and $E(\Delta R)$ can be written as:

$$\Delta R = p \min \{ \xi, y \} - h(y - \xi)^+ - K\delta(y - x) - c(y - x), \quad (14)$$

$$E(\Delta R) = \int [p \min \{ \xi, y \} - h(y - \xi)^+] \phi(\xi) d\xi - K\delta(y - x) - c(y - x) = L(y) + cx - K\delta(y - x), \quad (15)$$

Based on the bounds for y defined by Con (4), we use a function $B(R)$ to simplify the upper bound for ordering quantity.

$$B(R) = \max \left\{ 0, \frac{R - K}{c} \right\}, \quad (16)$$

The optimality equation changes to the following.

$$F_n(x, R) = \max_{x \leq y \leq x + B(R)} \{ L(y) + cx - K\delta(y - x) + \int F_{n+1}((y - \xi)^+, R + \Delta R) \phi(\xi) d\xi \}. \quad (17)$$

The aim is to maximize the expected cash increment $F_1(R_0, x_0)$ over the planning horizon given initial cash R_0 and inventory x_0 .

4. ORDERING POLICY DISCUSSION

Since optimality function $F_n(x, R)$ for the dynamic programming model is two-dimensional, it is difficult to prove its K-convexity or K-concavity. By fixing x or fixing R and view $F_n(x, R)$ as a function of single decision variable, $F_n(x, R)$ does not show K-convex or K-concave, nor does it show (C, K) -convex or (C, K) -concave. This can be easily confirmed by some numerical examples (we omit them here).

For our problem, we find some characteristics for the optimal policy. We illustrate this with two numerical examples.

Case 1. There are 20 periods, demand is stationary, with distribution $\Pr(6) = 0.95$, $\Pr(7) = 0.05$. Fixed ordering cost $K = 20$, unit variable ordering cost $c = 1$, unit holding cost $h = 1$, selling price $p = 5$, unit salvage value $\gamma = 0.5$, initial cash balance $R = 30$.

The optimal ordering quantities $Q^*(x, R)$ for different initial inventory x and R in the first period, are given by Table 1.

Case 2. There are 3 periods, demand is non-stationary and follow Poisson distribution. Expected demand for each period are 7, 3, 22. Fixed ordering cost $K = 20$, unit variable ordering cost $c = 1$, unit holding cost $h = 1$, selling price $p = 5$, unit salvage value $\gamma = 0.5$, initial cash balance $R = 30$.

The optimal ordering quantities $Q^*(x, R)$ for different initial inventory x and R in the first period, are given by Table 2.

As an illustration of how to read the above results, in Case 1, suppose the initial inventory x is 2, initial cash R is 35, then the optimal ordering quantity is 10 units.

Several ordering characteristics can be observed from Case 1 and Case 2.

- The optimal ordering policy is not the (s, S) type.
- When initial inventory is large or initial cash is small, optimal ordering quantity is always zero. For example, in Case 1, when $x \geq 4$ or $R \leq 25$, $Q^* = 0$; in Case 2, when $x \geq 2$ or $R \leq 27$, $Q^* = 0$. Furthermore, the bound for R is different for different initial inventory. In Case 1, when $x = 0$, it is optimal not to order when $R \leq 25$; when $x = 1$ or 2 or 3, it is optimal not to order when $R \leq 27$; in Case 2, when $x = 0$, it is optimal not to order when $R \leq 27$; when $x = 1$, it is optimal not to order when $R \leq 28$.
- For Case 2, when initial cash is large, it is also optimal not to order. This bound is also related with initial inventory. For example, when $x = 0$, if $R \geq 40$, $Q^* = 0$; when $x = 1$, if $R \geq 33$, $Q^* = 0$;
- When ordering, it is optimal for the retailer to order as close to some inventory levels as possible. For a fixed initial inventory, there may be several those

order-up-to levels. We show the different order-up-to levels by drawing a vertical line segment below the ordering quantity values. For example, in Case 1, the order-up-to levels are 6, 12 and 18; in Case 2, the order-up-to levels are 10 and 11.

5. CONCLUSION

Cash flow management is very important to the survival of many small businesses. In this paper, we consider a cash constrained retailer maximizing its cash increments in a finite planning horizon. After building a stochastic dynamic programming model for this problem, we find several characteristics about the optimal inventory controlling pattern.

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Table 1. Case 1

		<i>R</i>																	
		21	23	25	27	29	31	33	35	37	39	41	43	45	47	49	51	53	55
<i>x</i>	0	0	0	0	6	6	11	12	12	12	18	18	18	18	18	18	18	18	18
	1	0	0	0	0	9	11	11	11	17	17	17	17	17	17	17	17	17	17
	2	0	0	0	0	9	10	10	10	16	16	16	16	16	16	16	16	16	16
	3	0	0	0	0	9	9	9	9	9	9	9	9	9	9	9	9	9	9
	4	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	5	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	6	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

Table 2. Case 2

		<i>R</i>																	
		25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40	41	42
<i>x</i>	0	0	0	7	8	9	10	11	12	12	12	0	0	0	0	0	0	0	0
	1	0	0	0	0	9	9	10	9	0	0	0	0	0	0	0	0	0	0
	2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	3	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	4	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	5	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	6	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

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